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CONTENTS

Mathematics. — *On the geometry of spin spaces.* II. By J. A. SCHOUTEN.

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§ 4. *The quantities R, C, Ω and Π in $\mathfrak{S}_N$.*

In order to determine the subgroup of the affine group in $\mathfrak{S}_N$ corresponding to G_{or} in R_n we have to find first its invariants.

Next to the ordinary vectors v^A in $\mathfrak{S}_N$ we introduce the *vectors of the second kind* $\bar{u}^{\bar{A}}$ by their transformation

$$\bar{u}^{\bar{A}'} = \bar{A}^{\bar{A}'}_{\bar{A}}\, \bar{u}^{\bar{A}} \quad ; \quad \bar{A}^{\bar{A}'}_{\bar{A}} \stackrel{\text{def}}{=} \overline{A^{A'}_{A}} \qquad \bar{A} = \bar{1}, \ldots, \bar{N};\ \bar{A}' = \bar{1}', \ldots, \bar{N}' \quad . \quad (4.1)$$

Then the complex conjugates $\bar{v}^{\bar{A}}$ of the components of any ordinary vector v^A are components of a vector of the second kind. In the same way to every affinor e.g. $P^{A}_{.BC}$ corresponds an affinor of the second kind $\bar{P}^{\bar{A}}_{.\bar{B}\bar{C}} \stackrel{\text{def}}{=} \overline{P^{A}_{.BC}}$. There are also quantities with indices with and without a bar, called *hybrid quantities*, e.g.

$$Q^{\bar{A}'B'} = \bar{A}^{\bar{A}'}_{\bar{A}}\, A^{B'}_{B}\, Q^{\bar{A}B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

To every quantity of valence 2 belongs a quantity with the transposed matrix, called the *transposed* and denoted by a dash *to the left* of the kernel. E.g.

$$\left.\begin{array}{lll} 'P_{B}^{.A} \stackrel{\text{def}}{=} P^{A}_{.B} & ; & 'Q_{AB} \stackrel{\text{def}}{=} Q_{BA} \\ 'R^{\bar{A}}_{.B} \stackrel{\text{def}}{=} R_{B}^{.\bar{A}} & & 'S^{A\bar{B}} = S^{\bar{B}A} . \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4.3)$$

If for an ordinary co- or contravariant quantity, e.g. T_{AB} the invariant equation

$$'T_{AB} = \pm\, T_{AB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.4)$$

holds, it is called a *tensor* or *bivector* respectively. In the same way $U_{\bar{A}B}$ could be called a *hybrid tensor* or *bivector* if in the invariant equation

$$'\bar{U}_{\bar{A}B} = \pm\, U_{\bar{A}B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.5)$$

the $+$- or $-$-sign holds. But it is usual to call a hybrid tensor and also its matrix *hermitian*. From the definition it follows that if $U_{\bar{A}B}$ is a hybrid bivector, $i\,U_{\bar{A}B}$ is hermitian. This is the reason why hybrid bivectors are not often mentioned. If for a mixed hybrid quantity $V^{\bar{A}}_{.B}$ with $\mathrm{Det}\,(V^{\bar{A}}_{.B}) \neq 0$ the invariant equation

$$V^{\bar{A}}_{.B} = \pm \overset{-1}{\bar{V}}{}^{\bar{A}}_{.B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.6)$$

holds, $V^{\bar{A}}_{.B}$ is called *positive* or *negative invertible*. Multiplication with a factor of the form $e^{i\varphi}$ does not change this property. All properties

mentioned here are invariant for all *real and complex* coordinate transformations. But as to matrices we remark that the property of being symmetrical or alternating, hermitian or antihermitian and positive or negative invertible is invariant only if the matrix belongs to a co- or contravariant ordinary quantity, a co- or contravariant hybrid quantity or a mixed hybrid quantity respectively. In the following these properties of matrices will only be used if they have an invariant meaning. This is not the case in many physical publications, where a definite coordinatesystem in $\mathfrak{S}_N$ is preferred and not changed during the investigation. Of course this is quite correct because in this case the question of invariance for coordinate transformations can not arise. It is also correct to make use of properties as symmetry etc. that only exist with respect to some definitely given representation provided that the point of view taken is stated very clearly.

To every hermitian quantity $P_{\bar{A}B}$ (or $Q^{\bar{A}B}$) there always exists at least one real or complex coordinatesystem such that the matrix of the components with respect to this system has the diagonal form with the numbers $-1, -1, \ldots, -1$ (s times), $+1, \ldots, +1$ ($r-s$ times), $0, \ldots, 0$ in the maindiagonal. The sequence $-1, \ldots, 0$ is called the *signature*, r the *rank* and s the *index* of the quantity. The proof is wellknown.

To every positive or negative invertible quantity $P^{\bar{A}}_{\cdot B}$ there always exists at least one real or complex coordinatesystem such that the matrix of the components with respect to this system has the diagonal form with only numbers $+1$ in the maindiagonal in the positive case or the form

$$\left\| \begin{matrix} 0 & +1 & & & \\ -1 & 0 & & & \\ & & 0 & +1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{matrix} \right\| \quad \ldots\ldots\ldots \quad (4.7)$$

in the negative case. In order to prove this JACOBSON [1]) proved that if

$$P\bar{P} = Q\bar{Q} = \pm 1 \quad \ldots\ldots\ldots \quad (4.8)$$

there always exists a transformation S such that

$$\bar{S} Q S = P. \quad \ldots\ldots\ldots \quad (4.9)$$

In fact, if we take

$$S = \bar{P}(e^{i\theta} P + e^{-i\theta} Q) = \pm e^{i\theta} + e^{-i\theta} \bar{P} Q \quad \ldots \quad (4.10)$$

we have

$$S^{-1} = (e^{i\theta} P + e^{-i\theta} Q)^{-1} \bar{P}^{-1}. \quad \ldots\ldots \quad (4.11)$$

and

$$\left.\begin{aligned} \bar{S} Q S^{-1} &= (\pm e^{-i\theta} + e^{i\theta} P \bar{Q})\, Q\, (e^{i\theta} P + e^{-i\theta} Q)^{-1} \bar{P}^{-1} = \\ &= \pm (e^{-i\theta} Q + e^{i\theta} P)(e^{i\theta} P + e^{-i\theta} Q)^{-1} \bar{P}^{-1} = P. \end{aligned}\right\} (4.12)$$

[1]) Cf. VEBLEN and GIVENS, l.c. p. 4. 23. There seems to be a mistake in the formulae corresponding to (4. 10) and (4. 11).

Next to the Clifford set $\underset{j}{i}{}^{A}_{.B}$ we consider now several other Clifford sets, all with the same signature $\underset{1}{\varepsilon}{}^2, \ldots, \underset{n}{\varepsilon}{}^2$.

1°. The set $-\underset{j}{i}$. According to theorem II there exists an automorphism $S \ldots S^{-1}$ such that

$$-\underset{j}{i} = S \underset{j}{i} S^{-1}. \quad (4.13)$$

If S is normalized by the condition $S S = A$ we see that $S = R$ (R is determined to within the sign).

2°. The set of numbers $h^{A}_{.B}$ of C_n having the same matrices as the $'i_{A}^{.B}$ (mark the place of the indices!).

$$\underset{j}{h}{}^{A}_{.B} = \underset{j}{'i}{}_{A}^{.B} = \underset{j}{i}{}^{.B}_{.A}. \quad (4.14)$$

This set having the same multiplication rules as the set $\underset{j}{i}$, according to theorem II there exists an automorphism such that

$$S^{A}_{.C} \underset{j}{i}{}^{C}_{.D} \overset{-1}{S}{}^{D}_{.B} = \underset{j}{h}{}^{A}_{.B} \left(= \underset{j}{'i}{}_{A}^{.B}\right). \quad (4.15)$$

or in another form

$$C_{BC} \underset{j}{i}{}^{C}_{.D} \overset{-1}{C}{}^{DA} = \underset{j}{'i}{}_{B}^{.A} \quad (4.16)$$

or shortly

$$C \underset{j}{i} C^{-1} = \underset{j}{'i}. \quad (4.17)$$

Taking the transposed of both sides we get

$$'C^{-1} \underset{j}{'i} 'C = \underset{j}{i} \quad \text{or} \quad 'C \underset{j}{i} 'C^{-1} = \underset{j}{'i}. \quad (4.18)$$

from which we see that $'C :: C$ and this is only possible if $'C = \pm C$, that is, if C is either a tensor or a bivector. C is determined to within a scalar factor. Det (C_{AB}) is a scalar density of weight $+2$ [2]). Hence

$$|\mathrm{Det}\,(C_{AB})|^{-1/N} C_{AB} \quad (4.19)$$

is a tensor- or bivector-W-density of weight $-2/N$ determined to within a scalar factor of the form $e^{i\varphi}$. In the following we denote *this* quantity by C_{AB} or C. Then $|\mathrm{Det}\,(C_{AB})| = +1$. It is easily proved that

$$a)\ C \underset{j_1 \ldots j_p}{i} C^{-1} = (-1)^{\binom{p}{2}} \underset{j_1 \ldots j_p}{'i}\ ; \quad b)\ C R C^{-1} = (-1)^{\nu}\, 'R. \quad (4.20)$$

---

[2]) A scalar density of *weight* $\mathfrak{k}$ and *antiweight* $\mathfrak{k}'$ is a quantity that gets a factor $\underset{s}{\Delta}{}^{-\mathfrak{k}}\, \underset{s}{\bar{\Delta}}{}^{-\mathfrak{k}'}$ where $\underset{s}{\Delta}$ is the determinant of the transformation of coordinates in $\mathfrak{S}_N$. If $\mathfrak{k} = \mathfrak{k}'$ the quantity is also called a W-density of weight $2\,\mathfrak{k}$. The term antiweight was introduced by VEBLEN. W-densities are named after WEYL. An affinor-density or -W-density is the product of an affinor with a density or W-density respectively.

3°. The set of numbers of C_n having the same matrices as the $\underset{j}{{}'\bar{i}}{}^{\cdot\bar{A}}_{B}$. Because of

$$\underset{j}{{}'\bar{i}}\,\underset{k}{{}'\bar{i}} = \begin{cases} \underset{j}{\bar{\varepsilon}^2} = \underset{j}{\varepsilon^2} & \text{for } j=k \\ -\underset{k}{{}'\bar{i}}\,\underset{j}{{}'\bar{i}} & ,,\ \ j\neq k \end{cases} \quad . \quad . \quad . \quad . \quad . \quad (4.21)$$

this set is really a Clifford set with the same signature as the set $\underset{j}{i}$. Hence there exists a quantity $\Omega_{\bar{A}B}$ such that

$$\underset{j}{{}'\bar{i}} = \Omega\,\underset{j}{i}\,\Omega^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.22)$$

From (4.22) it follows that

$$\underset{j}{i} = {}'\bar{\Omega}^{-1}\,\underset{j}{{}'\bar{i}}\,{}'\bar{\Omega} \quad ; \quad \underset{j}{{}'\bar{i}} = {}'\bar{\Omega}\,\underset{j}{i}\,{}'\bar{\Omega}^{-1}. \quad . \quad . \quad . \quad . \quad (4.23)$$

hence

$${}'\bar{\Omega} = a\,\Omega \quad ; \quad a\bar{a} = +1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.24)$$

If Ω is multiplied by a scalar β we have

$$\bar{\beta}\,{}'\bar{\Omega} = a\frac{\bar{\beta}}{\beta}\beta\,\Omega, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.25)$$

hence β can always be chosen such that $\beta\,\Omega$ is hermitian. Then $\beta\,\Omega$ is determined to within a *real* scalar factor. The determinant of *this* $\beta\,\Omega$ is a real scalar-W-density of weight $+2$. Hence, if in

$$|\,\mathrm{Det}\,(\beta\,\Omega)|^{-1/N}\,\beta\,\Omega$$

we take in the denominator one of the two *real* values, this expression is a hermitian tensor-W-density of weight $-2/N$ with determinant ± 1. For this quantity, that is determined to within the sign, we write in the following $\Omega_{\bar{A}B}$ or Ω. Then $\mathrm{Det}(\Omega_{\bar{A}B}) = \pm 1$. It is easily proved that

$$a)\ \ \Omega\,\underset{j_1\ldots j_p}{i}\,\Omega^{-1} = (-1)^{\binom{p}{2}}\,\underset{j_1\ldots j_p}{{}'\bar{i}} \quad ; \quad b)\ \ \Omega R\,\Omega^{-1} = (-1)^{r}\,{}'\bar{R} \quad (4.26)$$

4°. The set of numbers of C_n having the same matrices as the $\underset{j}{\bar{i}}{}^{\bar{A}}_{\cdot B}$ leads in the same way to a hybrid quantity $\Pi^{\bar{A}}_{\cdot B}$ that can be normalized at once by taking

$$\Pi = {}'\Omega^{-1}\,C = \bar{\Omega}^{-1}\,C\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.27)$$

making use of the circumstance that we can first transform $\underset{j}{i}$ into $\underset{j}{{}'i}$ and than $\underset{j}{{}'i}$ into $\underset{j}{\bar{i}}$. But we could as well first transform $\underset{j}{i}$ into $\underset{j}{{}'\bar{i}}$ and then $\underset{j}{{}'\bar{i}}$ into $\underset{j}{\bar{i}}$. Hence

$$\Pi = \pm\,\bar{C}^{-1}\,\Omega = \pm\,\bar{C}^{-1}\,{}'\bar{\Omega}\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.28)$$

But from (4.27, 28) it follows that

$$\Pi^{-1} = \pm\,\Omega^{-1}\,\bar{C} = \pm\,\bar{\Pi}\ \ . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.29)$$

which means that Π is *positive or negative invertible.*

From the other possibilities only quantities arise that are comitants of R, C and Ω, e.g. [3])

5^0. The $-{}'i_{j}{}_B{}^A$ lead to $\mathfrak{R} \overset{\text{def}}{=} CR = \pm {}'R'C = \pm {}'\mathfrak{R}$. (4. 30)

6^0. The $-{}'\bar{i}_{j}{}_{\bar{B}}{}^{\bar{A}}$ lead to $\Omega R = \pm {}'\bar{R}\Omega$ . . . . (4. 31)

7^0. The $-i_{j}{}^{\bar{A}}{}_{\bar{B}}$ lead to $\Pi R = \pm \bar{R}\Pi$ . . . . (4. 32)

All these quantities are W-densities with a weight $+1/N$ or $-1/N$ for every upper or lower index respectively. Transvection and multiplication of these quantities leads always to quantities with this same property. Accordingly in the sequel we only use this kind of quantities in $\mathfrak{S}_N$ and call them *spinors* (*spinvectors*, *spintensors* etc.). Then the components of spinvectors transform with a determinant with modulus $+1$. Hence it comes to the same as if we had chosen from the beginning for $\mathfrak{S}_N$ the space belonging to the subgroup of the affine group with $|\underset{s}{\triangle}| = +1$ [4]). In this space there is no difference between the W-densities used here and ordinary quantities and the word density needs not to be used [5]).

Now we have to investigate the invariance of R, C and Ω for the transformations of $\mathfrak{S}_N$ corresponding to the transformations G_{or} of R_n. According to (2. 12) R is invariant for all rotations and gets a factor -1 for all reflexotations.

If $\underset{j}{i} \rightarrow \underset{j'}{i}$ is a real or complex transformation of G_{or}, according to theorem II there exists a transformation T in $\mathfrak{S}_N$ with $|\operatorname{Det}(T)| = +1$ such that

$$\underset{j'}{i} = T \underset{j}{i} T^{-1} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4.33)$$

T is determined to within a factor of the form $e^{i\varphi}$. Now, according to (4. 22) and (4. 33) we have for *real* orthogonal transformations

$$'\underset{j'}{\bar{i}} = \Omega \underset{j'}{i} \Omega^{-1} = \Omega T \underset{j}{i} T^{-1} \Omega^{-1} \quad . \; . \; . \; . \; . \; . \quad (4.34)$$

because the $\underset{j'}{i}$, depend linearly on the $\underset{j}{i}$ with *real* coefficients. But at the other hand it follows from the same equations that

$$'\underset{j'}{\bar{i}} = {}'\bar{T}^{-1} \, '\underset{j}{\bar{i}} \, '\bar{T} = {}'\bar{T}^{-1} \Omega \underset{j}{i} \Omega^{-1} \, '\bar{T}, \quad . \; . \; . \; . \quad (4.35)$$

hence

$$'\bar{T} \Omega T = \lambda \Omega \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4.36)$$

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[3]) There are other possibilities, e.g. $\underset{j}{i} \rightarrow I \underset{j}{i}$.

[4]) By means of C this subgroup will be reduced later to the subgroup with $\underset{s}{\triangle} = \pm 1$.

[5]) This clears the mystery why those authors who treat the wave vector φ^A in spinspace as a vector meet no difficulties and come to the same results as those who consider φ^A as a vector-density of weight $1/N$. In some way (often not explicity) the first mentioned authors have introduced the condition $|\underset{s}{\triangle}| = +1$.

where λ is a scalar coefficient. From (4.36) it follows that

$$\overline{T}'\overline{\Omega}T=\lambda'\overline{\Omega} \text{ or } '\overline{T}\Omega T=\overline{\lambda}\Omega \quad . \quad . \quad . \quad . \quad . \quad (4.37)$$

Hence λ is *real* and from $|\mathrm{Det}\ (T)|=+1$, $|\mathrm{Det}\ (\Omega)|=+1$ it follows that $\lambda=\pm 1$. That proves that Ω is invariant to within the sign for *real* orthogonal transformations.

For all real or complex orthogonal transformations it follows from (4.17) and (4.33) at one hand

$$'\underset{j'}{i}=C\underset{j'}{i}C^{-1}=CT\underset{j}{i}T^{-1}C^{-1} \quad . \quad . \quad . \quad . \quad . \quad (4.38)$$

and at the other hand

$$'\underset{j'}{i}='T^{-1}\,'\underset{j}{i}\,'T='T^{-1}C\underset{j}{i}C^{-1}\,'T, \quad . \quad . \quad . \quad . \quad (4.39)$$

hence

$$'TCT=\mu C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.40)$$

where μ is a scalar factor of the form $e^{i\varphi}$. If T gets a factor β, from

$$\beta'TC\beta T=\mu\beta^2 C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.41)$$

we see that β can be chosen in such a way that $\mu\beta^2=+1$. Writing T for *this* new value βT we have *normalized* T but for the sign and obtained that C is invariant for all normalized transformations in T corresponding to the real or complex transformations of G_{or}. According to

$$'TCT=C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.42)$$

we see that now $\mathrm{Det}\ (T)=\pm 1$.

We recall that p linearly independent contravariant vectors given in a definite order fix a p-sense ($=p$-dimensional screwsense) in the E_p spanned by these vectors, and that a p-sense and an $(n-p)$-sense in an E_p and an E_{n-p} in E_n fix an n-sense, provided that they are given in a definite order and that the E_p and the E_{n-p} have no direction in common. Be $\underset{1}{i},\ldots,\underset{s}{i}$ mutually orthogonal real unit vectors in the $-$-region of an R_n with index s. If an n-sense is given the s-sense of the i's fixes an $(n-s)$-sense in *every* real R_{n-s} in the $+$-region. Accordingly, if in the $+$-region only a real R_{n-s} with an $(n-s)$-sense is given, a definite $(n-s)$-sense is induced in *every* real R_{n-s} in the $+$-region. We call these $(n-s)$-senses *the same* and the other $(n-s)$-sense *opposite*. The same holds for an R_s with an s-sense in the $-$-region. Now we consider a *real* orthogonal transformation $\underset{1}{i},\ldots,\underset{n}{i}\rightarrow\underset{1'}{i},\ldots,\underset{n'}{i}$. Then $\underset{1'}{i},\ldots,\underset{s'}{i}$ span an R_s with an s-sense in the $-$-region and $\underset{(s+1)'}{i},\ldots,\underset{n'}{i}$ an R_{n-s} with an $(n-s)$-sense in the $+$-region. The transformation will be called *$+$-reflexional* if the two $(n-s)$-senses are opposite and *$-$-reflexional* if the two s-senses are opposite. This gives four possible cases [6])

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[6]) Cf. for the algebraical conditions BRAUER and WEYL l.c. p. 441 f.

a)	rotation ; $\Delta = +1$	not $+$-refl.	not $-$-refl.	(4.43)
b)	reflexotation ; $\Delta = -1$	$+$-refl.	not $-$-refl.	
c)	reflexotation ; $\Delta = -1$	not $+$-refl.	$-$-refl.	
d)	rotation ; $\Delta = +1$	$+$-refl.	$-$-refl.	

In the manifold of all *real* transformations of G_{or} the transformations of the same kind can be transformed into each other in a continuous way [7]) but transformations of different kinds cannot. This follows immediately from the fact that an n-sense in E_n cannot be transformed continuously into the opposite n-sense. Hence the coefficient λ in (4.36), that can have the values $+1$ and -1, must have the same value for all orthogonal transformations of the same kind. The transformation $\underset{1}{\varepsilon}\,\underset{1}{i}\,R$ transforms $\underset{1}{i}$ into $-\underset{1}{i}$ and leaves $\underset{2}{i}, \ldots, \underset{n}{i}$ invariant:

$$\left.\begin{aligned} \underset{1}{\varepsilon}\underset{1}{i}R\underset{1}{i}R\underset{1}{\varepsilon}\underset{1}{i} &= -\underset{1}{i} \\ \underset{1}{\varepsilon}\underset{1}{i}R\underset{2}{i}R\underset{1}{\varepsilon}\underset{1}{i} &= +\underset{2}{i} \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (4.44)$$

and accordingly represents a reflexion at the R_{n-1} perpendicular to $\underset{1}{i}$. Hence $\underset{j}{\varepsilon}\underset{j}{i}R$ is in the case (b) for $\underset{j}{\varepsilon}^2 = +1$ and in the case (c) for $\underset{j}{\varepsilon}^2 = -1$. According to (4.20$b$) and (4.17)

$$\underset{j}{\varepsilon}'\underset{j}{i}'RCR\underset{j}{\varepsilon}\underset{j}{i} = (-1)^r \underset{j}{\varepsilon}^2\,'\underset{j}{i}\,C\,\underset{j}{i} = (-1)^r C, \quad \ldots \quad (4.45)$$

hence the normalized transformation is $i^r \underset{j}{\varepsilon}\underset{j}{i}R$. By this normalized transformation Ω transforms into

$$\left.\begin{aligned} &(-i)^r \bar{\underset{j}{\varepsilon}}\,'\bar{R}\,'\bar{\underset{j}{i}}\,\Omega\, i^r \underset{j}{\varepsilon}\underset{j}{i}R = '\bar{R}\,'\bar{\underset{j}{i}}\,\Omega\,\underset{j}{i}R = \\ &= '\bar{R}\,'\bar{\underset{j}{i}}\,'\bar{\underset{j}{i}}\,\Omega R = \underset{j}{\varepsilon}^2\,'\bar{R}\,\Omega R = \\ &= \underset{j}{\varepsilon}^2\,'\bar{\underset{n}{i}}\ldots'\bar{\underset{1}{i}}\,\Omega\,\underset{1}{i}\ldots\underset{n}{i} = \underset{j}{\varepsilon}^2\underset{1}{\varepsilon}^2\ldots\underset{n}{\varepsilon}^2\,\Omega = (-1)^s \underset{j}{\varepsilon}^2\,\Omega \end{aligned}\right\} . \quad (4.46)$$

according to (4.22). From this we see that Ω changes its sign if and only if s is odd and $\underset{j}{\varepsilon}\underset{j}{i}R$ is $+$-reflexional or if s is even and $\underset{j}{\varepsilon}\underset{j}{i}R$ is $-$-reflexional. The cases (a) and (d) can be obtained by repeated application of transformations (b) and (c). That gives the table.

Case	s even: Ω	s even: ΩR	s odd: Ω	s odd: ΩR		
(a)	Ω	ΩR	Ω	ΩR	neither $+$- nor $-$-refl.	(4.47)
(b)	Ω	$-\Omega R$	$-\Omega$	ΩR	$+$-refl. but not $-$-refl.	
(c)	$-\Omega$	ΩR	Ω	$-\Omega R$	$-$-refl. but not $+$-refl.	
(d)	$-\Omega$	$-\Omega R$	$-\Omega$	$-\Omega R$	$+$-refl. and $-$-refl.	

[7]) Cf. CARTAN, l.c. I p. 18.

If $v^{\varkappa_1 \dots \varkappa_p}, p = s$ or $p = n - s$ is a simple real p-vector in the —- or +-region respectively, its area, measured by a parallelotop of p mutually perpendicular unit vectors is the positive number

$$(\pm 1)^p \, 1/p! \; v^{\varkappa_1 \dots \varkappa_p} v_{\varkappa_1 \dots \varkappa_p} \begin{cases} \text{for } p = n - s \\ \text{for } p = s \end{cases} \quad . \quad . \quad . \quad (4.48)$$

If $w^{\varkappa_1 \dots \varkappa_p}$ is another simple real p-vector in the same region it has the same p-sense as $v^{\varkappa_1 \dots \varkappa_p}$ if

$$(\pm 1)^p \, 1/p! \; v^{\varkappa_1 \dots \varkappa_p} w_{\varkappa_1 \dots \varkappa_p} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.49)$$

is > 0 and the opposite p-sense in the other case. Now take $p = s$ and

$$v^{\varkappa_1 \dots \varkappa_p} = p! \; \underset{1}{i}^{[\varkappa_1} \dots \underset{s}{i}^{\varkappa_s]} \quad ; \quad w^{\varkappa_1 \dots \varkappa_p} = p! \; \underset{1'}{i}^{[\varkappa_1} \dots \underset{s'}{i}^{\varkappa_s]}$$

then (4.49) takes the form

$$(-1)^s \begin{vmatrix} \underset{1'}{i}^{\varkappa} \underset{1}{i}_{\varkappa} & \dots & \underset{1'}{i}^{\varkappa} \underset{s}{i}_{\varkappa} \\ \underset{s'}{i}^{\varkappa} \underset{1}{i}_{\varkappa} & & \underset{s'}{i}^{\varkappa} \underset{s}{i}_{\varkappa} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.50)$$

Now if T is the transformation $\underset{j}{i} \rightarrow \underset{j'}{i}$ and if $T^{h}_{.i}$ are its orthogonal components with respect to the $\underset{j}{i}$, we have

$$\underset{j'}{i}^{h} = T^{h}_{.i} \underset{j}{i}^{i} \overset{*}{=} T^{h}_{.j} \quad ; \quad h, i, j = 1, \dots, n \quad ; \quad j' = 1', \dots, n' \quad (4.51)$$

or

$$T^{h}_{.j} \overset{*}{=} \underset{j'}{i}^{\varkappa} A^{h}_{\varkappa} \overset{*}{=} \underset{j'}{i}^{\varkappa} \overset{h}{i}_{\varkappa} \overset{*}{=} \varepsilon^2 \underset{h}{} \underset{j'}{i}^{\varkappa} \underset{h}{i}_{\varkappa} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.52)$$

and accordingly

$$T^{a}_{.b} \overset{*}{=} - \underset{b'}{i}^{\varkappa} \underset{a}{i}_{\varkappa} \quad ; \quad a, b = 1, \dots, s \quad ; \quad b' = 1', \dots, s' \quad . \quad (4.53)$$

Hence the necessary and sufficient condition that the transformation T is not —-reflexional takes the form

$$\operatorname{Det}(T^{a}_{.b}) > 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.54)$$

In the same way it is proved that T is +-reflexional if and only if $\operatorname{Det}(T^{x}_{.y}) < 0$; $x, y = s + 1, \dots, n$ [8]).

We collect results in the following table for the determinant, the undetermined factor and the invariance with respect to *normalized* transformations of G_s, (see following page).

From the invariant quantities R, C, Ω and Π of course R was from the beginning known to all investigators. C seems to have appeared first for

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[8]) This form of the conditions is due to BRAUER and WEYL, l.c. p. 441. Moreover these authors proved that both subdeterminants are always ≤ -1 or $\geq +1$. For $s = 3$, $n = 4$ this is the wellknown theorem of shortening of moving bars and retardation of moving clocks. The proof is easy but we do not need this result in the sequel.

	Det.	untermined factor	invariant to within factor			
			a	b	c	d
R	-1 for $\nu=1$ $+1$ for $\nu>1$	± 1	$+1$	-1	-1	$+1$
C	mod. $=+1$	$e^{i\varphi}$	$+1$	$+1$	$+1$	$+1$
Ω	± 1	± 1	$+1$	$(-1)^s$	$(-1)^{s+1}$	-1
Π	mod. $=+1$	$e^{i\varphi}$	$+1$	$(-1)^s$	$(-1)^{s+1}$	-1
T (normalized)	± 1	± 1				

$\nu = 2$ in our paper of 1931 [9]). Ω was derived in our paper of 1933 [10]) for $\nu = 2$ and at the same time by VEBLEN [11]) and also by PAULI [12]). PAULI's method, that he applied also to C and that in this paper is applied to all invariant quantities, is based upon the theorem II of BRAUER and WEYL, though at that time this theorem was not yet formulated explicitly. As a consequence at that time it seemed still necessary to prove the invariance of Ω, as we did in 1933 [13]) for $\nu = 2$ and real rotations at the instigation of a correspondence with PAULI. As PAULI remarked [14]) Ω was already introduced by BARGMANN in 1932, though only for a special case. The form $\bar{\varphi}\Omega\varphi$ was already known to FOCK in 1929 [15]). The quantity Π, being derivable from C and Ω, appeared for $\nu = 2$ already in VEBLEN's paper of 1933 and in our paper of 1935 [16]) on conformal fieldtheory and for general values of ν in the VEBLEN–GIVENS lectures of 1935—36. From the projective point of view of these authors R stands for an involution, C for a polarity or null-polarity, Ω for an anti-polarity and Π for an anti-involution. But it is quite possible that there exist still other priority claims. As to some notations by other authors, CARTAN's C is equal to our C for ν even and to K for ν odd. BRAUER and WEYL's C is equal to our C but their B is equal to our Ω for s even and equal to ΩR for s odd.

(To be continued.)

[9]) Dirac equations in general relativity, Journ. of Math. and Phys. **10** (1931) 240—283, p. 262.

[10]) Generelle Feldtheorie V, Raumzeit und Spinraum, Z. f. Physik **81** (1933) 405—417.

[11]) Spinors in projective relativity, Proc. Nat. Acad. **19** (1933) 979—989.

[12]) Ueber die Formulierung der Naturgesetze mit fünf homogenen Koordinaten, Ann. d. Phys. **18** (1933) 305—336, 337—372.

[13]) L.c. p. 414.

[14]) L.c. p. 347.

[15]) Geometrisierung der Dirac'schen Theorie des Elektrons, Z. f. Phys. **57** (1929) 261—277; L'équation d'onde de Dirac et la géométric de RIEMANN, J. de Ph. et le R. (6) **10** (1929) 392—405.

[16]) J. A. SCHOUTEN and J. HAANTJES, Konforme Feldtheorie II, R_6 und Spinraum, Ann. di Pisa, II, **4** (1935) 175—189.

Mathematics. — *The second pearl of the theory of numbers.* By J. G. VAN DER CORPUT and J. H. B. KEMPERMAN. (First communication.)

(Communicated at the meeting of June 25, 1949.)

§ 1. *The $A+B$-theorem.*

These communications contain theorems with or without weights. If the set A consists of real numbers, we denote in the theorems without weights by $A(h)$ the number of elements $< h$ of A. Let $\psi(a)$ be a real function defined for every element a of A; we denote by $\psi(A)$ the set of numbers, which may be written in at least one way as $\psi(a)$, where a is an element of A, for instance $2A$, A^2, and (if each element of A is positive) $\log A$.

If B is also a set formed by real numbers, we denote by the sum $A+B$ of A and B the set of numbers, which may be written in at least one way as the sum $a+b$ of two terms belonging respectively to A and B. In a similar manner AB denotes the set formed by the numbers ab. In the special case that A and B denote the same set, we find that $A+A$ resp. AA is the set of numbers, which may be written in at least one way as the sum resp. product of two numbers, both belonging to A, whereas $2A$ (resp. A^2) is the set of numbers $2a$ (resp. a^2), where a is an arbitrary element of A. For instance $(A^3+BB)(h)$ denotes in the theorems without weights the number of numbers $< h$, which may be written in at least one way in the form a^3+bb', where a is an element of A, and b, b' are elements of B.

It is to be noted that in the definition of $A(h)$ the number h is not counted, even if it belongs to A. We prefer this definition to the conventional one, in which the element h, if it belongs to A, is counted in $A(h)$, and where the number 0 is never counted, even if it belongs to A. In fact our definition gives many simplifications in the enunciation of our results.

Between 1930 and 1935 the mathematicians SCHNIRELMANN, KHINTCHINE and LANDAU have stated the following $\alpha+\beta$-hypothesis:

Let A and B be two sets of non-negative integers, both containing the number 0; let g be a positive integer, α and β two real numbers such that $\alpha+\beta \leqq 1$. Then the inequalities

$$\sum_{0<a\leq h} 1 \geqq \alpha h \qquad (h=1,2,\dots g) \quad \dots \dots \quad (1)$$

and

$$\sum_{0<b\leq h} 1 \geqq \beta h \qquad (h=1,2,\dots g) \quad \dots \dots \quad (2)$$

imply

$$\sum_{0<a+b\leq h} 1 \geqq (\alpha+\beta) h \qquad (h=1,2,\dots g). \quad \dots \dots \quad (3)$$

(1) means that for each positive integer $h \leqq g$ the number of positive elements $a \leqq h$ of A is at least equal to αh; similarly (3) means that for

each positive integer $h \leq g$ the number of positive integers $\leq h$, which may be written in at least one way as a sum $a + b$, where a denotes an element of A and b an element of B, is equal to or greater than $(\alpha + \beta)h$.

Using the above definition of $A(h)$, the $\alpha + \beta$-theorem may also be formulated in the following form:

Theorem 1: *Let g be a positive integer, A and B two sets of integers ≥ 0, both containing 0, such that*

$$A(h) \geq 1 + \alpha(h-1); \quad B(h) \geq 1 + \beta(h-1) \quad (h = 1, 2, \ldots g),$$

where $\alpha + \beta \leq 1$. Then we have

$$(A + B)(h) \geq 1 + (\alpha + \beta)(h-1) \quad (h = 1, 2, \ldots, g).$$

Several mathematicians have devoted their efforts to prove theorem 1, till in 1942 H. B. MANN [1]) found a very ingenious proof. He obtained even more, viz.

Theorem 2: *Let g be a positive integer, A and B sets of numbers ≥ 0, both containing 0, such that*

$$A(h) + B(h) \geq 2 + \gamma(h-1) \quad (h = 1, 2, \ldots, g),$$

where $\gamma \leq 1$. Then

$$(A + B)(h) \geq 1 + \gamma(h-1) \quad (h = 1, 2, \ldots, g).$$

Theorem 1 is a special case of MANN's proposition (put $\alpha + \beta = \gamma$). Though the latter theorem lies deeper than the first, its proof is simpler, because the principle of induction may here be applied with success, whereas such an application is more difficult or impossible, if we start from the original version of the $\alpha + \beta$-theorem.

A. YA. KHINTCHINE has treated MANN's theorem as second problem in his book: Three pearls of the theory of numbers [2]).

The result found by MANN has been generalised and the argument has been simplified by E. ARTIN and P. SCHERK [3]), by F. J. DYSON [4]) and by J. G. VAN DER CORPUT [5]).

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[1]) H. B. MANN, A proof of the fundamental theorem on the density of sums of sets of positive integers, Annals of Math., **43**, 523—529 (1942).

[2]) A. YA. KHINTCHINE, Three pearls of the theory of numbers (Russian), Moscow 1947, 72 p. The first pearl is the problem of the boxes, stated as hypothesis by P. J. H. BAUDET and proved by B. L. VAN DER WAERDEN: if we put all positive integers in a finite number of boxes, then at least one of these boxes contains an arithmetical progression of thousand terms (we may replace thousand by any positive integer). The third pearl is the problem of Waring.

[3]) E. ARTIN and P. SCHERK, On the sum of two sets of integers, Annals of Math., **44**, 138—142 (1943).

[4]) F. J. DYSON, A theorem on the densities of sets of integers, Journal of the London Math. Society **20**, 8—15 (1945).

[5]) J. G. VAN DER CORPUT, On sets of integers, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 252—261, 340—350, 429—435 (1947). The reader may find the same articles in Indagationes Mathematicae, **9**, 159—168, 198—208, 257—263 (1947).

Let G be an ordered set such that the sum $g + g'$ of any two elements g and g' of G is defined and belongs to G. We say that $\varphi(h)$ increases slowly on a subset H of G, if $\varphi(h)$ is defined for all elements h of H such that

$$\varphi(h_3) \leqq \varphi(h_1) + \varphi(h_2),$$

where h_1, h_2 and h_3 denote arbitrary elements of H with $h_3 \leqq h_1 + h_2$. Thus $\varphi(h)$ is also increasing slowly on every subset of H.

If we take in the following proposition for H the set of all integers $\geqq 0$, we obtain the special case, which has been proved by J. G. VAN DER CORPUT [6]).

Theorem 3 (The $A + B$-theorem): *Let G be a set formed by numbers $\geqq 0$ such that the sum $g + g'$ of any two elements g and g' of G belongs to G. Let $\varphi(g)$ be slowly increasing on the set formed by the positive elements of G and let H be a subset of G.*

If the finite subsets A and B of H satisfy for each positive element h of H the inequality

$$A(h) + B(h) \geqq 1 + \varphi(h), \qquad (4)$$

then we have for these elements h also

$$(A + B)(h) \geqq \varphi(h). \qquad (5)$$

Remark: If $\varphi(h)$ is a monotonic non-decreasing function on the set, formed by the positive elements of G, and if (4) is true for $h = g$ and for each positive number $< g$ belonging to A or B, where g denotes a given positive element of G, then (4) (and therefore under the conditions of theorem 3 also (5)) is true for each positive element $h \leqq g$ of G. In fact, if h' denotes the smallest number $\geqq h$, which is equal to g or belongs to A or B, we have

$$A(h) + B(h) = A(h') + B(h') \geqq 1 + \varphi(h') \geqq 1 + \varphi(h).$$

If we take in the $A + B$-theorem for G the set of the integers $\geqq 0$, for H the set $1, 2, \ldots, g$ and for $\varphi(h)$ the slowly increasing function $1 + \gamma(h-1)$, where $\gamma \leqq 1$, we obtain theorem 2, since the elements $> g$ of A and B do not enter into consideration.

If g is a positive number and the finite sets A and B, consisting of numbers $\geqq 0$, satisfy for $h = g$ and for each positive number $h < g$ belonging to A or B, the inequalities

$$A(h) \geqq \alpha h + \gamma \quad \text{and} \quad B(h) \geqq \alpha h + \gamma - \tfrac{1}{2}, \qquad (6)$$

where $\alpha \geqq 0$ and $\gamma \geqq \frac{1}{2}$, then

$$(A + B)(h) \geqq 2\alpha h + 2\gamma - 1,$$

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[6]) Loco citato, theorem 2; Proceedings p. 261 and Indagationes p. 168.
E. TROST repeated this proof in concise form on one page in Elemente der Math. 2 (1947), p. 103.

for each positive number $h \leqq g$. Indeed, we have the identity

$$\{u\} + \{u - \tfrac{1}{2}\} = \{2u\} \geqq 2u,$$

where $\{u\}$ denotes the smallest integer $\geqq u$, so that the assertion follows from the preceding theorem, applied with the monotonic function $\varphi(h) = 2ah + 2\gamma - 1$.

A. YA. KHINTCHINE [7]) has found for the set H of positive integers the special case, in which $\frac{1}{2} \leqq \gamma \leqq 1 - \alpha$ and $\alpha < \frac{1}{2}$ and where (6) is replaced by the somewhat stronger conditions

$$A(h) \geqq \alpha h + 1 - \alpha \text{ and } B(h) \geqq \alpha h + \gamma - \tfrac{1}{2} \quad (h = 1, \ldots, g).$$

His proof is important, because it is the first proof of the special case $\alpha = \beta$ of theorem 1.

The $A + B$-theorem contains

The AB-theorem: *Let G be a set formed by numbers $\geqq 1$ such that the product gg' of any two elements g and g' of G belongs to G. Let $\varphi(g)$ be defined for each element $g > 1$ of G such that*

$$\varphi(g_3) \leqq \varphi(g_1) + \varphi(g_2),$$

where g_1, g_2, g_3 denote arbitrary elements > 1 of G with $g_3 \leqq g_1 g_2$.

If H is a subset of G and if the finite subsets A and B of H satisfy for each element $h > 1$ of H the inequality

$$A(h) + B(h) \geqq 1 + \varphi(h),$$

then we have for these elements h also

$$AB(h) \geqq \varphi(h).$$

For the proof of the AB-theorem it is sufficient to replace G, H, A, B, g, h and 1 respectively by $\log G$, $\log H$, $\log A$, $\log B$, $\log g$, $\log h$ and 0.

We may apply the AB-theorem if $\varphi(h)$ is a slowly increasing function of $\log h$, for instance $\varphi(h) = \gamma \log h + \delta$, where $\gamma \geqq 0$ and $\delta \geqq 0$.

The $A + B$-theorem gives also

The $A^k + B^k$-theorem: *Let k and g be positive numbers and let $\varphi(h)$ be slowly increasing on the set formed by the positive numbers.*

If the finite sets A and B, formed by numbers $\geqq 0$, satisfy for each positive number $h \leqq g$ the inequality

$$A(h) + B(h) \geqq 1 + \varphi(h^k),$$

then

$$(A^k + B^k)(h) \geqq \varphi(h)$$

for each positive number $h \leqq g^k$.

Indeed the elements $> g$ of A and B do not enter into consideration, so that it is sufficient for the proof to apply the $A + B$-theorem, where G

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7) A. YA. KHINTCHINE, Zur additiven Zahlentheorie, Matematiceski Sbornik, **39**, 27—34 (1932).

denotes the set of the numbers $\geqq 0$ and H the set of numbers $\geqq 0$ and $\leqq g^k$ and to replace in the $A+B$-theorem A, B, h and g respectively by A^k, B^k, h^k and g^k.

By putting $k=2$ and taking for $\varphi(h)$ the slowly increasing function $(\alpha+\beta)\sqrt{h}$, where $\alpha+\beta\geqq 0$, we obtain in particular: if the finite sets A and B, formed by numbers $\geqq 0$, satisfy for each positive number $h \leqq g$ the inequalities

$$A(h) \geqq \theta + \alpha h \quad \text{and} \quad B(h) \geqq 1 - \theta + \beta h,$$

where $g > 0$, then we have for these numbers h also

$$(A^2+B^2)(h) \geqq (\alpha+\beta)\sqrt{h}.$$

This proposition gives a rather sharp inequality for sets A and B containing many elements below a given bound. This is not so for the analoguous theorems involving sets as AA or $AA+BB$. In order to give an impression of the results connected with sets of that kind, we give an example.

Let A and B be two finite sets of integers $\geqq 0$. If for $\alpha \geqq 0$ and $\beta \geqq 0$

$$A(h) \geqq \tfrac{3}{2} + \alpha \log h, \qquad B(h) \geqq \tfrac{3}{2} + \beta \log h$$

for $h = 2, 3, \ldots, g$, where g denotes an integer $\geqq 2$ and $\alpha+\beta \leqq \dfrac{1}{\log 4}$, then we have for the positive integers $\leqq g$

$$(AA+BB)(h) \geqq 1 + 2(\alpha+\beta)\log h.$$

In fact the given inequalities furnish $A(2) \geqq \frac{3}{2}$ and $B(2) \geqq \frac{3}{2}$. Hence both A and B contain 0 and 1. The system A^* formed by the positive elements of A, satisfies for $h = 2, 3, \ldots, g$

$$A^*(h) \geqq \tfrac{1}{2} + \alpha \log h$$

and consequently by the AB-theorem

$$A^*A^*(h) \geqq 2\alpha \log h \qquad (h = 2, 3, \ldots, g),$$

hence

$$AA(h) \geqq 1 + 2\alpha \log h \qquad (h = 1, 2, \ldots, g).$$

We obtain a similar inequality for BB, hence

$$AA(h) + BB(h) \geqq 1 + \varphi(h) \qquad (h = 1, 2, \ldots, g),$$

where the monotonic function

$$\varphi(h) = 1 + 2(\alpha+\beta)\log h$$

satisfies for every pair of positive integers h and h' the inequality

$$\varphi(h)+\varphi(h')-\varphi(h+h') = 1-2(\alpha+\beta)\log\left(\frac{1}{h}+\frac{1}{h'}\right) \geqq 1-2(\alpha+\beta)\log 2 \geqq 0$$

in virtue of $\alpha+\beta \leqq \dfrac{1}{2\log 2}$. Consequently $\varphi(h)$ increases slowly on the set of the positive integers, so that the $A+B$-theorem applied with AA and BB in stead of A and B gives the assertion.

We give as last example:

Suppose that $\varphi(h)$ and $\chi(h)$ are slowly increasing monotonic non-decreasing functions for $h > 0$. Let g be a positive number and A and B two finite sets of numbers ≥ 0. Suppose

$$A(h) \geqq \theta + \varphi(h^3) \quad \ldots \ldots \ldots \quad (7)$$

for $h = g^{\frac{1}{3}}$ and for each positive element $h < g^{\frac{1}{3}}$ of A, and finally

$$B(h) \geq 1 - \theta + \chi(h^2)$$

for $h = g^{\frac{1}{2}}$ and for each positive element $h < g^{\frac{1}{2}}$ of B. Then we have for each positive number $h \leq g$.

$$(A^3 + B^2)(h) \geq \varphi(h) + \chi(h).$$

For the proof we remark, that (7) is valid for every positive number $h \leq g^{\frac{1}{3}}$, since $\varphi(h)$ is a monotonic non-decreasing function. Then we have for each positive number $h \leq g$

$$A^3(h) = A(h^{\frac{1}{3}}) \geq \theta + \varphi(h)$$

and similarly

$$B^2(h) = B(h^{\frac{1}{2}}) \geq 1 - \theta + \chi(h),$$

hence

$$A^3(h) + B^2(h) \geq 1 + \varphi(h) + \chi(h)$$

where $\psi(h) = \varphi(h) + \chi(h)$ increases slowly for $h > 0$. Consequently the $A + B$-theorem gives the assertion.

The $A + B$-theorem may be generalised in the following manner on sets belonging to abstract algebra:

Theorem 4: *Let G be an ordered* [8]) *set, containing a smallest element denoted by 0, on which a commutative and associative addition has been defined with $g + 0 = g$ and $g + g^* > g$ for $g^* > 0$, such that*

$$g + g' = g + g'' \text{ implies } g' = g''. \quad \ldots \ldots \quad (8)$$

Let $\varphi(g)$ be slowly increasing on the set formed by the elements > 0 of G and let H be a subset of G.

If the finite subsets A and B of H, containing both the element 0, satisfy for each positive element h of H the inequality

$$A(h) + B(h) \geqq 1 + \varphi(h), \quad \ldots \ldots \ldots \quad (4)$$

then we have for these elements h also

$$(A + B)(h) \geqq \varphi(h). \quad \ldots \ldots \ldots \quad (5)$$

The remark following immediately after the $A + B$-theorem is here also true.

That the $A + B$-theorem is a special case of theorem 4, is obvious, except if the number 0 does not belong to both sets A and B, but in this last case we have $\varphi(h) \leq 0$ for every positive element h of H (so that then

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[8]) "Ordered" means in these communications always that the order is transitive.

the assertion of the $A + B$-theorem is evident). In fact, if neither A nor B contains a positive number, we have

$$1 \geqq A(h) + B(h) \geqq 1 + \varphi(h),$$

for every positive element h of H, and otherwise we find for the smallest positive number h^* belonging to A or B

$$1 \geqq A(h^*) + B(h^*) \geqq 1 + \varphi(h^*);$$

the slowly increasing function $\varphi(h)$ satisfies therefore the inequalities

$$\varphi(h^*) \leqq 0; \ \varphi((n+1)h^*) \leqq \varphi(nh^*) + \varphi(h^*) \leqq 0$$

$(n = 1, 2, \ldots)$, and

$$\varphi(h) \leqq \varphi(nh^*) + \varphi(h^*) \leqq 0$$

if n is chosen so large that $h \leqq nh^* + h^*$.

Theorem 4 is obvious if 0 is the only element of B, for in that case we find for each positive element h of H

$$B(h) = 1; \ A(h) \geqq \varphi(h); \ (A+B)(h) = A(h) \geqq \varphi(h).$$

In the proof of theorem 4 we will therefore suppose that B contains at least one positive element. The smallest element $\bar{a}$ of A, such that B contains at least one positive element b with the property that $\bar{a} + b$ does not belong to A, is called by us the basic element of the couple A and B. Such an element exists, since the greatest element a of A has the property that $a + b$, where b is an arbitrary positive element of B, is greater then a and consequently does not belong to A.

Lemma 1. *Under the conditions of theorem 4 we have*

$$A(a) \geqq \varphi(a)$$

for each positive element $a \leqq \bar{a}$ of A, when $\bar{a}$ denotes the basic element of the pair A and B.

Proof: Suppose $\bar{a} > 0$, for otherwise there is nothing to be proved. The smallest positive element b of B satisfies the relation $B(b) = 1$ and therefore also

$$A(b) \geqq \varphi(b). \qquad (9)$$

The assertion of lemma 1 is obvious for the positive elements $a \leqq b$ of A, because then

$$B(a) = 1; \ A(a) + 1 = A(a) + B(a) \geqq 1 + \varphi(a).$$

In the proof we may therefore suppose $b < a \leqq \bar{a}$. We will deduce the inequality in question for $a = a_1$, where a_1 denotes an element $> b$ and $\leqq \bar{a}$ of A, under the assumption that the inequality has already been proved for all positive elements $a < a_1$ of A.

The element $0 < a_1$ belongs to A, so that A contains a greatest element $a_2 < a_1$.

In virtue of $a_2 < a_1 \leqq \bar{a}$ and the minimum property of the basic element $\bar{a}$

the element $a_2 + b$ is an element $> a_2$ of A, and therefore $\geqq a_1$. The inequalities

$$a_3 < a_1 \leqq a_3 + b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

are satisfied, if a_3 is replaced by a_2, so that A contains a smallest element a_3 with (10). Then $a_3 \leqq a_2 < \bar{a}$. By the minimum property of the basic element each element $a < a_3$ has therefore the property that $a + b$ belongs to A and this element is by the minimum property of a_3 less then a_1. From (8) it follows that different elements $a < a_3$ furnish different elements $a + b$ and these elements $a + b$ are $\geqq b$ and $< a_1$. Consequently

$$A(a_1) - A(b) \geqq A(a_3).$$

We have $b < a_1 \leqq a_3 + b$, hence $a_3 > 0$. In virtue of $a_3 < a_1$ we find by induction

$$A(a_3) \geqq \varphi(a_3)$$

and consequently

$$\begin{aligned} A(a_1) \geqq A(b) + A(a_3) &\geqq \varphi(b) + \varphi(a_3) && \text{by (9)} \\ &\geqq \varphi(a_1) && \text{by (10)}, \end{aligned}$$

since $\varphi(h)$ increases slowly. This establishes the proof of lemma 1.

Let us denote by $b_1, \ldots, b_k$ the positive elements b of B such that $\bar{a} + b$ does not belong to A. From the definition of the basic element $\bar{a}$ it follows that $k \geqq 1$. By canceling b_1 we transform B into a set B_1; by canceling b_2 we transform B_1 into a set $B_2, \ldots$, by canceling finally b_k we transform B_{k-1} into a set B_k. By adding $\bar{a} + b_1$ we transform A and H into the sets A_1 and H_1; by adding $\bar{a} + b_2$ we transform A_1 and H_1 into the sets A_2 and H_2; ..., by adding finally $\bar{a} + b_k$ we transform A_{k-1} and H_{k-1} into the sets A_k and H_k. It is obvious that A_k and B_k are subsets of H_k.

It is easy to see that $A_k + B_k$ is a subset of $A + B$. In fact, an arbitrary element h of $A_k + B_k$ has the form $h = a^* + b^*$, where a^* belongs to A_k and b^* belongs to B_k. If a^* is an element of A, then h is an element of $A + B$. If on the contrary a^* does not belong to A, it is added to A in the construction of A_k, so that it has the form $a^* = \bar{a} + b_j$; as b^* has not been canceled in the construction of B_k, it has the property that $a = \bar{a} + b^*$ belongs to A; consequently

$$h = a^* + b^* = (\bar{a} + b_j) + b^* = (\bar{a} + b^*) + b_j = a + b$$

belongs to $A + B$.

Lemma 2: *Under the conditions of theorem 4 we have*

$$A_l(h) + B_l(h) \geqq 1 + \varphi(h)$$

for each positive element h of H_l and for $l = 1, 2, \ldots k$.

Proof. If $l \geqq 2$ we may assume

$$A_{l-1}(h) + B_{l-1}(h) \geqq 1 + \varphi(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

for each element h of H_{l-1}. This inequality holds also for $l = 1$, if A_0, B_0 and H_0 denote the sets A, B and H.

For the positieve elements $h \leqq b_l$ of H_l we have

$$B_{l-1}(h) = B_l(h) \quad \text{and} \quad A_{l-1}(h) \leqq A_l(h)$$

and for the elements $h > \bar{a} + b_l$ of H_l we have

$$A_{l-1}(h) + 1 = A_l(h) \quad \text{and} \quad B_{l-1}(h) \leqq B_l(h) + 1,$$

so that the assertion follows for these elements from (11), provided that h belongs to H_{l-1}. This is indeed so, for otherwise h would be the element $\bar{a} + b_l$, added to H_{l-1}, and from $h \leqq b_l$ or $> \bar{a} + b_l$ would follow $\bar{a} = 0$ so that h would coincide with the element b_l, which belongs to B and therefore to H, consequently also to H_{l-1}.

We may therefore suppose that h is an element of H_l with

$$b_l < h \leqq \bar{a} + b_l. \quad \ldots \ldots \ldots \quad (12)$$

The inequality

$$a_4 + b_l \geqq h \quad \ldots \ldots \ldots \quad (13)$$

is true, if a_4 is replaced by $\bar{a}$. The smallest element a_4 of A with (13) is therefore $\leqq \bar{a}$. From the minimum properties of a_4 and $\bar{a}$ it follows for each element $a < a_4$ that $a + b_l$ is less then h and belongs to A. Different elements $a < a_4$ of A furnish by (8) different elements $a + b_l$ of A and these elements are $\geqq b_l$ and $< h$. Consequently

$$A(h) - A(b_l) \geqq A(a_4)$$

and therefore, since A is a subset of A_l,

$$A_l(h) - A_l(b_l) \geqq A(a_4)$$

From $b_l < h \leqq a_4 + b_l$ it follows that $a_4 > 0$, so that we deduce from $a_4 \leqq \bar{a}$ by lemma 1

$$A(a_4) \geqq \varphi(a_4).$$

In this manner we find, as b_l is a positive element $< h$ of H_{l-1}

$$\begin{aligned} A_l(h) + B_l(h) &\geqq A(a_4) + A_l(b_l) + B_l(h) \\ &\geqq \varphi(a_4) + A_{l-1}(b_l) + B_{l-1}(b_l) \\ &\geqq \varphi(a_4) + 1 + \varphi(b_l) && \text{by} \quad (11) \\ &\geqq 1 + \varphi(h) && \text{by} \quad (13). \end{aligned}$$

End of the proof of theorem 4.

We may suppose that the number $n+1$ of elements of B is > 1, for the theorem is otherwise obvious. We may also suppose that the theorem has already been proved, if n is replaced by a smaller integer $\geqq 0$. By lemma 2 we see, that the conditions of theorem 4 remain valid, if the sets A, B and H are replaced by the sets A_k, B_k and H_k. Since B_k contains less then $n+1$ elements we obtain by induction

$$(A_k + B_k)(h) \geqq \varphi(h)$$

for the positive elements h of H_k. This establishes the proof, as $A_k + B_k$ is a subset of $A + B$, and H is a subset of H_k.

Mathematics. — *On the symbolical method.* III. By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of May 28, 1949.)

The relations obtained in the foregoing part:

$p_{iklmnp} + p_{imlpnk} + p_{iplknm} = 0$, STEINERpoint-incidence;

$p_{kmplin} + p_{pkminl} + p_{mpknli} = 0$, KIRKMANpoint-incidence;

$\langle kmplin\rangle\, p_{kmplin} + \langle pkminl\rangle\, p_{pkminl} + \langle mpknli\rangle\, p_{mpknli} \equiv$
$\equiv \langle iknmlp\rangle\, p_{iknmlp} + \langle imnplk\rangle\, p_{imnplk} + \langle ipnklm\rangle\, p_{ipnklm} \equiv c_{iln,kmp}$;

$K_{iklmnp} + K_{imlpnk} + K_{iplknm} = 0$, CAYLEY-SALMON-incidence;

$K_{kmplin} + K_{pkminl} + K_{mpknli} \equiv 2\, S_{kmp,\, iln}$;

$\langle iklmnp\rangle\, K_{iklmnp} + \langle imlpnk\rangle\, K_{imlpnk} + \langle iplknm\rangle\, K_{iplknm} \equiv$
$\equiv \langle inl\rangle\, \langle kmp\rangle\, S_{inl,kmp}$;

show, that the system of these points and lines is, from the algebraical point of view, to be completed by the point on the line p_{iklmnp}:

$$B_{iklmnp} \equiv \langle kmplin\rangle\, K_{kmplin} + \langle mpknli\rangle\, K_{mpknli} + \langle pkminl\rangle\, K_{pkminl} = 0.$$

Evidently $B_{iklmnp} \equiv B_{klmnpi} \equiv - B_{pnmlki}$, so there are 60 points B_{iklmnp}. These points are the points of intersection of p_{iklmnp} with the harmonical lines of K_{iklmnp} with regard to the triangles of the VERONESE-decade of which K_{iklmnp} is the centre of perspectivity.

Intersection of the lines

$$p_{iklnmp} \equiv (li)\,(pk)\, a_n\, a_m - (ln)\,(pm)\, a_i\, a_k = 0,$$
$$p_{mlnpki} \equiv (nm)\,(il)\, a_k\, a_p - (nk)\,(ip)\, a_m\, a_l = 0$$

gives:

$$\begin{aligned} &(li)\,(pk)\,(nm)\,(il)\,(nk)\, a_m\, a_p + (li)\,(pk)\,(nm)\,(il)\,(mp)\, a_n\, a_k + \\ &- (li)\,(pk)\,(nk)\,(ip)\,(nl)\, a_m^2 - (ln)\,(pm)\,(nm)\,(il)\,(ip)\, a_k^2 \\ &+ (ln)\,(pm)\,(nk)\,(ip)\,(il)\, a_k\, a_m + (ln)\,(pm)\,(nk)\,(ip)\,(km)\, a_i\, a_l \equiv \\ &\equiv \langle mplin\rangle\, a_k^2 + \langle kmpln\rangle\, a_i^2 + (li)^2\, \langle nmk\rangle\, a_p^2 - \langle kmiln\rangle\, a_p^2 \equiv \\ &\equiv \langle kmplin\rangle\, K_{kmplin}. \end{aligned}$$

Thus B_{iklmnp} is the point of intersection of $p_{iklmnp} = 0$ and

$b_{iklmnp} \equiv p_{kimlnp} + p_{ikmlpn} + p_{kilmpn} = 0$ or of $p_{iklmnp} = 0$ and
$b_{klmnpi} \equiv p_{iknmpi} + p_{klnmip} + p_{lkmnip} = 0$

from which we have

$$b_{iklmnp} - b_{klmnpi} \equiv 3\, p_{iklmnp}.$$

Now K_{iklmnp} is the centre of perspectivity of the triangles

$$K_{mpilkn},\ K_{klpinm},\ K_{lnmipk} \text{ and}$$
$$K_{lmpnki},\ K_{pnikml},\ K_{mikpnl}.$$

The last triangle can be obtained from the first by cyclical permutation of $iklmnp$. The lines $b_{iklmnp} = 0$ and $b_{klmnpi} = 0$ are the harmonical lines of $K_{iklmnp} = 0$ with regard to these triangles. Indeed, the pencil of lines through the point of intersection of the fourth harmonical line of

$$K_{iklmnp}\ K_{mpilkn} \equiv p_{pkminl}$$

with regard to the other PASCAL-lines through K_{mpilkn} and the line $K_{klpinm}\ K_{lnmipk} \equiv p_{kilmpn}$ is

$$p_{kinplm} + p_{kimlnp} + \lambda\, p_{kilmpn} = 0.$$

The analogous relation for the point K_{klpinm} gives

$$p_{kilmpn} + p_{kimlnp} + \mu\, p_{kinplm} = 0$$

from which the harmonical line results for $\lambda = \mu = 1$ i.e. $b_{iklmnp} = 0$. The 120 lines b_{iklmnp} meet by two in the points B_{iklmnp} on p_{iklmnp}.

§ 6. *Representation of pencils of quantics.*

For the discussion of the degenerations in Hexagramma mysticum, obtained from a non-degenerated hexagon, we need some properties of pencils of quadratic forms, which can easily be generalised.

The points of intersection of a conic and the curves of order n of a pencil having k of its basis-points on the conic represent a pencil of quantics of order $2n - k$; the common tangents of a conic and the curves of class n of a pencil touch the conic in points representing a pencil of quantics of order $2n - k$ if k of the basis-tangents of the pencil are tangents of the conic.

Proof: Be the curve of order n $(a'X)^n = 0$, then the points of intersection with $(AU')a_t^2 = 0$ are

$$(a'A)(a'B)\ldots(a'C)\, a_t^2\, b_t^2 \ldots c_t^2 = \varphi_t^{2n} = 0.$$

If $a'_{\ldots} \rightarrow a'_{\ldots} + \lambda\, b'_{\ldots}$ we obtain a pencil of quantics of order $2n$, which have a factor of order k, independent of λ, in common if k of the basis-points of the pencil are on the conic. q.e.d.

The dual theorem and the generalisation to normal-curves are self-evident.

For $n = 1$, we obtain the condition that the lines $[ik] = 0$, $[lm] = 0$, $[np] = 0$ are concurrent from the simultaneous invariant of three binary quadratic forms to be

$$R(ik, lm, np) = (ip)(lk)(nm) - (im)(lp)(nk) = 0.$$

The cases in which $k > 1$ lead to generalisations of the theorem of concurrence of the cords of three circles:

a. If two basis-points of a pencil of curves of order n are on a conic the point-sets of intersection formed by the remaining points are the intersection of the conic with a pencil of curves of order $n-1$.

b. If $2n-2$ basis-points of a pencil of curves of order n are on a conic the lines joining the remaining points are concurrent.

c. If $2n-3$ basis-points of a pencil of curves of order n are on a conic the remaining three points of intersection with the curves form a triangle, the sides of which are tangent to a conic, for:

Be 1, 2, 3 and i, k, l two triangles inscribed in the conic, the equation of the conic touching the six sides can easily be found. In ternary symbols the equation is

$$\lambda(1\,U')(2\,U')+\mu(2\,U')(3\,U')+\nu(3\,U')(1\,U')=0$$

and the condition that the line P_iP_k is a tangent becomes, splitting up the ternary brackets and dividing by $(ik)^2$

$$\lambda(1\,i)(1\,k)(2\,i)(2\,k)+\mu(2\,i)(2\,k)(3\,i)(3\,k)+\nu(3\,i)(3\,k)(1\,i)(1\,k)\equiv\Omega(i,k)=0.$$

Evidently $\Omega(i,k)=0$, $\Omega(k,l)=0$, $\Omega(l,i)=0$ are three equations in λ, μ, ν the determinant of which vanishes, demonstrating the fact that the six sides are tangents of the same conic. Solving $\lambda:\mu:\nu$ we have putting $(i\,k)(k\,x)(l\,x)=\varphi_x^3$

$$(\lambda,\mu,\nu)=(\varphi_3^3(12),\varphi_1^3(23),\varphi_2^3(31))$$

and the equation of the conic is, omitting the ternary factor $(AU')(BU')$:

$$\varphi_3^3(12)\,a_1^2\,b_2^2+\varphi_1^3(23)\,a_2^2\,b_3^2+\varphi_2^3(31)\,a_3^2\,b_1^2=0.$$

As the lefthand-side vanishes for $1\equiv 2$, $2\equiv 3$, $3\equiv 1$ we can divide by a factor $<123>$ and obtain putting $(1\,x)(2\,x)(3\,x)=\psi_x^3$

$$(KU')^2=[c_1(\varphi\psi)^3(ab)^2+c_2(\varphi\psi)(\varphi a)^2(\psi b)^2](AU')(BU')=0,$$

which shows that every pair of triples of the pencil $\varphi+\lambda\psi$ gives the same conic $(KU')^2=0$.

The equation of the conic having all the triangles of a pencil inscribed in a conic as polar-triangles can easily be found.

Putting in ternary symbols

$$L(12\,X)^2+M(23\,X)^2+N(31\,X)^2=0$$

we have for the condition that P_i and P_k are conjugated, breaking up the ternary brackets

$$L(12)^2(1i)(1k)(2i)(2k)+M(23)^2(2i)(2k)(3i)(3k)+N(31)^2(3i)(3k)(1i)(1k)=0,$$

which is the same equation in $L(12)^2$, $M(23)^2$, $N(31)^2$ as the $\Omega(i,k)=0$ in λ, μ, ν.

We have therefore, omitting the ternary factor $(A'X)(B'X)$:

$$\varphi_1^3(23)\,\alpha_2\,\alpha_3\,\beta_2\,\beta_3 + \varphi_2^3(31)\,\alpha_3\,\alpha_1\,\beta_3\,\beta_1 + \varphi_3^3(12)\,\alpha_1\,\alpha_2\,\beta_1\,\beta_2 = 0.$$

which vanishes for $1 \equiv 2$, $2 \equiv 3$, $3 \equiv 1$, and splitting off the factor $<123>$ we obtain

$$[c_3\,(\varphi\psi)^3(\alpha\beta)^2 + c_4\,(\varphi\psi)\,(\varphi a)^2(\psi b)^2]\,(A'X)\,(B'X) = 0.$$

The discriminant of $(KU')^2$ clearly vanishes only if the resultant of the two cubics φ_x^3, ψ_x^3 vanishes.

As regards the simultaneous invariants of $(KU')^2$ and $(\Omega'X)^2 = (A'X)(B'X)(ab)^2$, the invariant $(K\Omega')^2$ is apart from a factor I^2 an invariant of φ_x^3, ψ_x^3 linear both in φ and ψ so

$$(K\Omega')^2 = c_5\,I^2\,(\varphi\psi)^3.$$

$(K'\Omega)^2$ is the product of I^4 and an invariant of degree 2 both in φ and ψ, which can easily be obtained to be the square of $(\varphi\psi)^3$, apart from a constant:

The simultaneous invariants of K and Ω vanish for apolar cubics. This can be shown in a very elementary way.

If 12 and 45 are harmonical pairs than the condition that

$$(1x)\,(2x)\,(3x) = \varphi_x^3 \text{ and } (4x)\,(5x)\,(6x) = \psi_x^3$$

are apolar is $R(12, 45, 36) = 0$.

Proof:

$$6\,(\varphi\psi)^3 = (14)\,(26)\,(35) + (15)\,(26)\,(34) + (16)\,(24)\,(35) + \\ + (16)\,(25)\,(34) + (14)\,(25)\,(36) + (15)\,(24)\,(36) = 0.$$

The last two terms cancel as $(14)\,(25) + (15)\,(24) = 0$ and the identical transformations indicated give

$$6\,(\varphi\psi)^3 = 2\,(26)\,(14)\,(35) + \\ -(31)\,[(24)\,(65) + (25)\,(64) + (26)\,(54)] = 2\,R\,(12, 45, 36) = 0. \quad \text{q.e.d.}$$

If we now choose the isotropic points in 1,2 Ω becomes a circle, 45 a diameter and 36 a parallel to this diameter (fig. 1).

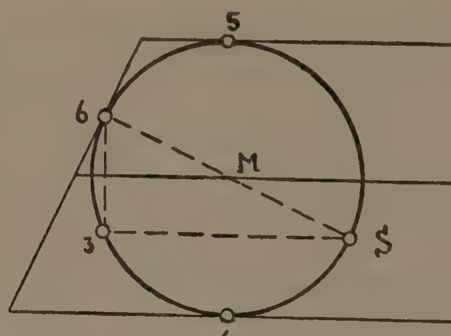

Fig. 1.

K is the parabola with focus in 3, tangent to the sides of 456, having therefore $6\,MS$ as directrix (64 and 65 being orthogonal tangents and S the image of 3 in the tangent 45).

As 63 is orthogonal to $3\,S$ the triangle $36\,S$ is a polar triangle of K *inscribed* in Ω.

Again: the tangent in 6 at Ω is perpendicular to the directrix and has therefore its pole in the point at infinity of $3\,S$. The second tangent through M at K is perpendicular to 45. So the triangle of tangents in 4, 5, 6 at Ω form a polar triangle of K *circumscribed* to Ω.

From this is seen how the study of point sets in a plane can be reduced to the theory of binary quadratic forms; that for three dimensional space to the theory of a binary quadratic and a system of cubic forms; and so on.

Representing the points of three dimensional space by a triple on a conic, the lines of space correspond to PONCELET-conics for triangles and the conics with equianharmonical intersection quadruples, for which both simultaneous invariants vanish form a line-complex which using a normal-curve in $C_4 : (AU')\,a_t^3$ corresponds to the complex $(A\,B\,\pi^2)\,(ab)^3 = 0$.

§ 7. *Degenerations in Hexagramma mysticum.*

A. $P_{ik,\,lm}$.

If two PASCAL-points coincide at least three sides of the hexagon are concurrent.

$$R(ik, lm, np) = 0,$$

and the invariant R of the sextic $\varphi_x^6 = 0$, defining the hexagon vanishes.

There are 15 different relations of this type.

a. If there exist two relations simultaneously having one couple in common e.g.

$$R(ik, lp, mn) \equiv (kn)\,(ml)\,(pi) + (np)\,(lk)\,(im) = 0$$
$$R(ik, ln, mp) \equiv (kp)\,(ml)\,(ni) - (np)\,(lk)\,(im) = 0$$

we can replace these taking the sum and dividing by (ml) by

$$(kn)\,(pi) + (kp)\,(ni) = 0;$$

taking the difference:

$$(ml)\,[(\mathbf{kn})\,(\mathbf{pi}) - (kp)\,(ni)] + 2\,(np)\,(lk)\,(im) \equiv (\mathbf{ml})\,(pn)\,(\mathbf{ki}) + 2\,(np)\,(lk)\,(im) = 0$$

or dividing by (pn)

$$(km)\,(li) + (kl)\,(mi) = 0.$$

The points i, k are the double points of the involution lm, np. *The hexagon is composed from a tetragon and one of its diagonals.*

b. If there exist two relations simultaneously having no couple in common there exists a third relation.

$$R(ik, lm, np) \equiv (ip)(lk)(nm) - (im)(lp)(nk)$$
$$R(im, lp, nk) \equiv (ik)(lm)(np) - (ip)(lk)(nm)$$
$$R(ip, lk, nm) \equiv (im)(lp)(nk) - (ik)(lm)(np)$$

shows $R(ik, lm, np) + R(im, lp, nk) + R(ip, lk, nm) \equiv 0$.

From the involutions

$$i \rightleftarrows k \quad , \quad l \rightleftarrows m \quad , \quad n \rightleftarrows p$$
$$i \rightleftarrows m \quad , \quad l \rightleftarrows p \quad , \quad n \rightleftarrows k$$

we have

$$i \to k \to n \qquad n \to p \to l \qquad l \to m \to i$$
$$k \to i \to m \qquad m \to l \to p \qquad p \to n \to k$$

The cycle *inl* is formed by a cyclic projectivity $k \to m \to p \to k$.

The hexagon is composed from two triples having the same Hessian.

c. If three relations exist simultaneously they evidently cannot have all three a couple in common: the hexagon would be degenerated.

There are two possible triples, apart from permutations of i, k, l, m, n, p.

1. $R(ik, lm, np) = 0$ $R(il, km, np) = 0$ $R(im, ln, kp) = 0$, only one pair with a couple in common;

2. $R(ik, lm, np) = 0$ $R(il, km, np) = 0$ $R(ik, ln, mp) = 0$, two pairs have a couple in common.

ad 1. If we put in the first case

$$i_x = n_x + \lambda p_x, \; m_x = n_x - \lambda p_x, \; k_x = n_x + \mu p_x, \; l_x = n_x - \mu p_x$$

(according to a) we obtain from the third relation

$$\lambda^2 + \mu^2 = 0.$$

$[np] = 0$, $[im] = 0$, $[kl] = 0$ form a polar-triangle. φ_x^6 *is an octaeder form*: $(\varphi, \varphi)^{(6)} \equiv 0$.

ad 2. In the second case we have also the relation $R(ip, ml, nk) = 0$ so $[np, im]$, $[np, lk]$, $[ik, lp]$, $[ik, nm]$, $[ml, in]$, $[ml, pk]$ are according to a harmonical pairs. φ_x^6 *is the product of* f_x^3 *with the cubic covariant* Q_x^3 $f_x^3 = (ix)(nx)(lx)$ $Q_x^3 = (kx)(mx)(px)$. Or with $k_x^4 = (\varphi, \varphi)^4$:

$$R = 0, \; (\varphi, k)^{(4)} \equiv 0, \; (\varphi, \varphi)^{(6)} \not\equiv 0.$$

B. p_{iklmnp}.

In this section we call non adjacent, non opposite indices simply separated indices.

I. If the indices of two Pascal-lines do not contain adjacent indices in common, there are at least two pairs of separated indices. Indeed in

$$p_{1ab4cd} = p_{1\alpha\beta 4\gamma\delta}$$

we must have $\alpha \neq a$, $\alpha \neq d$ so $\alpha = b = 2$ or $\alpha = c = 2$. Again in

$$p_{1a24cd} = p_{12\beta 4\gamma\delta}, \qquad \beta \neq a, \; \beta \neq c, \text{ so } \beta = 3 = d.$$
$$p_{1a24c3} = p_{1234\gamma\delta} \qquad \delta \neq a, \text{ so } \delta = c = 6$$
$$p_{152463} = p_{123456}.$$

The case $a = c = 2$ gives $p_{136425} = p_{123456}$ which is the same coincidence.

By the permutation $152463 \rightarrow 234561$ we obtain thus for the only coincidence not containing adjacent indices the type

$$p_{123456} \equiv p_{153624}.$$

This coincidence will take place if $P_{36,41}$ and $P_{15,62}$ are incident with p_{123456}.

The first condition is

$$(62)(34)(51) + (35)(61)(42) = 0,$$

the second

$$(35)(64)(16)(12) + (31)(62)(41)(56) = 0.$$

Eliminating 5: from the first condition

$$5_x = (62)(34)\,1_x - (61)(42)\,3_x$$

substituting in the second relation

$$(64)(12)(34) + (41)(62)(34) + (41)(42)(36) = 0.$$

The lefthand-side vanishes for $4 = 2$, so dividing by (42) we obtain

$$\left.\begin{array}{r}(61)(34) + (63)(14) = 0 \\ (62)(34)(51) + (35)(61)(42) = 0\end{array}\right\}$$

as the conditions for $p_{123456} \equiv p_{153624}$.

The first condition shows now that 64, 13 are harmonical pairs. The second condition defines a projectivity $2 \barwedge 5$ with the corresponding elements $6 \rightarrow 3,\ 4 \rightarrow 1,\ 3 \rightarrow 4$.

Starting with 1, 3, 4 we can construct 6. P being the pole of 13 and Q the intersection of [61] and [34] we have to take 5, 2 such that [65] and [32] intersect on PQ.

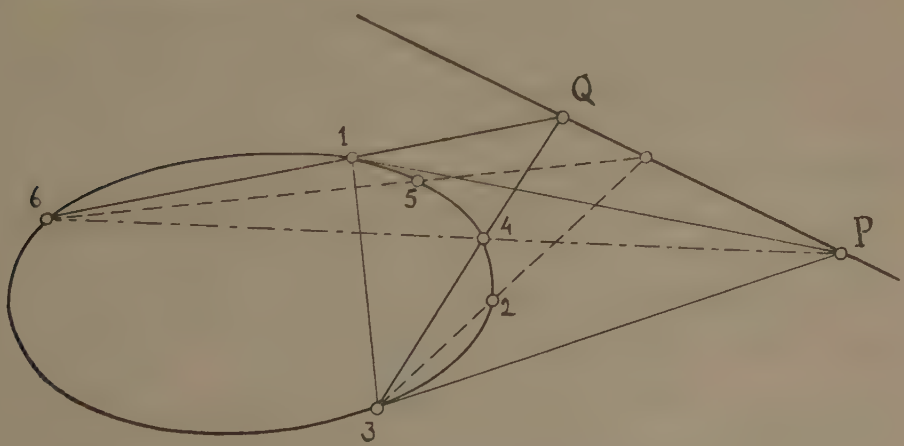

Fig. 2.

II. In all other cases we have a couple of adjacent indices in common

$$p_{123456} \equiv p_{12\alpha\beta\gamma\delta}.$$

This coincidence exists is $P_{12,\beta\gamma}$ and $P_{2\alpha,\gamma\delta}$ are incident with p_{123456}. The first condition gives

$$(1\beta)(42)(5\gamma) + (2\gamma)(51)(4\beta) = -R(12, 45, \beta\gamma) = 0$$

so either $(\beta, \gamma) \equiv (4, 5)$ or $(\beta, \gamma) = (3, 6)$ and [12], [45], [36] are concurrent.

The second condition gives

$$(31)(62)(2\gamma)[(5\alpha)(4\delta) + (5\delta)(4\alpha)] + (31)(62)(\alpha\delta)[(52)(4\gamma) + (5\gamma)(42)] +$$
$$-(35)(64)(2\gamma)[(1\alpha)(2\delta) + (1\delta)(2\alpha)] - (35)(64)(\alpha\delta)(12)(2\gamma) = 0, \text{ or}$$
$$(31)(62)(2\gamma)(5\delta)(4\alpha) + (31)(62)(\alpha\delta)(52)(4\gamma) - (35)(64)(2\gamma)(1\alpha)(2\delta) = 0.$$

1. If $(\beta, \gamma) \equiv (4, 5)$ and $\gamma = 4$ we have

$$\alpha = 3,\ \delta = 6 : \quad (31)(62)(24)(56)(43) - (35)(64)(24)(13)(26) = 0$$
$$\text{or} \qquad (56)(43) - (35)(64) \equiv (63)(54) = 0$$
$$\alpha = 6,\ \delta = 3 : \quad (31)(62)(24)(53)(46) - (35)(64)(24)(16)(23) = 0$$
$$\text{or} \qquad (31)(62) - (16)(23) \equiv (21)(63) = 0.$$

In both cases the hexagon must be degenerated and we are left with

$$\beta = 4,\ \gamma = 5,\ \alpha = 6,\ \delta = 3, \qquad p_{123456} \equiv p_{126453}.$$

We then have

$$(31)(62)(25)(53)(46) + (31)(62)(63)(52)(45) - (35)(64)(25)(16)(23) = 0,$$

which after division by (25) vanishes for $3 \equiv 6$ and can be divided by (36):

$$(13)(62)(45) + (35)(64)(21) = R(36, 25, 41) = 0.$$

2. If $(\beta, \gamma) \equiv (3, 6)$ we have the four cases

$$p_{123456} \equiv p_{124365},\ p_{123456} \equiv p_{125364},\ p_{123456} \equiv p_{124635},\ p_{123456} \equiv p_{125634}.$$

The second and third case are identical as is seen from the permutation $125364 \to 123456$, the first case gives a degenerated hexagon for from $\beta = 3$, $\gamma = 6$, $\alpha = 4$, $\delta = 5$ follows

$$(31)(62)(45)(52)(46) - (35)(64)(26)(14)(25) = 0 \text{ or}$$
$$(31)(45) + (35)(14) \equiv (43)(15) = 0.$$

The second case is

$$-(31)(62)(54)(45) + (31)(54)(52)(46) + (35)(64)(15)(24) = 0,$$

vanishing for $2 \equiv 4$, so

$$(31)(54)(65) + (35)(64)(15) = 0.$$

This relation is a projectivity $6 \to 1$ with $5 \to 5$, $4 \to 3$, $3 \to 4$. The projectivity is therefore an involution, the lines [16], [3, 4] and the

tangent in 5 being concurrent. Starting with the tangent in 5 and choosing [16], [34] concurrent with the tangent we find the point 2 from the concurrence of [12], [45], [36].

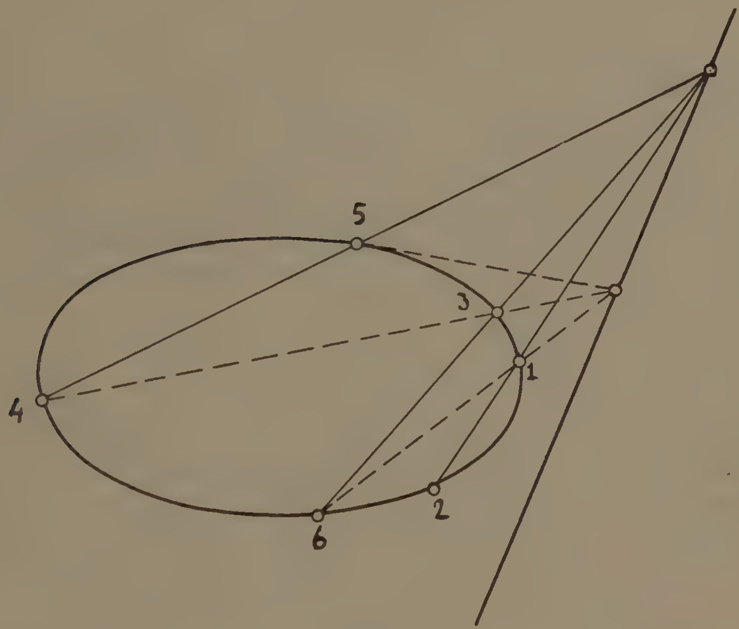

Fig. 3.

In the last case the STEINER-point $S_{135,246}$ is undetermined.
Indeed we have then:

$$(31)(62)(23)(54)(45)+(31)(62)(54)(52)(23)-(35)(64)(23)(15)(24) \equiv$$
$$(31)(62)(54)(42)(53)-(35)(64)(23)(15)(24)=0$$ or dividing by $(35)(42)$

$$R(16, 34, 52)=0,$$

together with

$$R(12, 36, 54)=0.$$

(To be continued.)

Mathematics. — *Affine embedding theory II: Frenet formulae.* By V. HLAVATÝ. (Communicated by Prof. J. A. SCHOUTEN.)

(Communicated at the meeting of February 26, 1949.)

Synopsis. This paper is a continuation of the previous one entitled *Affine embedding theory I* (Nederl. Akad. v. Wet., Vol. LII, No, 5, 1949) which will be referred to as *AI*. We shall find first two sets of η-tensors, K and L, and then establish the Frenet formulas where these tensors are involved. These formulas enable as to investigate the contact of two subspaces in A_n.

1. *The η-tensors L and K.*

Later on we shall need the following

Lemma (1, 1). *In a privileged parameter system* (cf. *AI Lemma* (4, 2)) *the coefficients* $p_{a'_x \ldots a'_r \ldots a'_1}^{\;a_r \ldots a_1}$ *which occur in. AI* (2, 3) *are*

$$p_{a'_x \ldots a'_r \ldots a'_1}^{\;a_r \ldots a_1} = P^{a_r \ldots a_1}_{b'_r \ldots b'_1} \, \omega_{a'_x \ldots a'_r \ldots a'_1}^{\;b'_r \ldots b'_1} \quad . \quad . \quad . \quad . \quad . \quad (1,1)$$

where ω *is a function of the* $\Phi_{a' \ldots}^{\;b' \ldots}$, *only.*

Proof. The γ's which appear in the equation of the footnote 7) in *AI* are expressed as a sum of products of the Γ's and moreover the ξ's are expressed as a sum of products of at least one of the Γ's with some of the $\Phi_{a' \ldots}^{\;b' \ldots}$'s (and eventually of some of the $P^{a}_{a'}$, $P^{a'}_{a}$). Hence the equation in the footnote 7) of *AI* reduces because of Lemma (4, 2) in *AI* to (1, 1), where the ω is a function of the $\Phi_{a' \ldots}^{\;b' \ldots}$ only.

Theorem (1, 1). *The elimination of the* $P^{b}_{a' \ldots}$ *from the equations AI* (2, 7 *a*, *b*) *and the equations AI* (4, 6), *AI* (3, 6 *b*) *leads to the transformation law for an η-tensor* $L_{a_{N+1} \ldots a_s \ldots a_1}^{\;b_s \ldots b_1}$, ($s = 2, \ldots, N$) *which has the following properties* :

a) *its components are given in a privileged parameter system* η^a *by*

$$L_{a_{N+1} \ldots a_s \ldots a_1}^{\;b_s \ldots b_1} = \Phi_{a_{N+1} \ldots a_s \ldots a_1}^{\;b_s \ldots b_1} \text{ at } P; \quad (s = 2, \ldots N) \quad . \quad (1,2)$$

b) *It is of the* $\left.\begin{matrix}\xi-\\ \eta-\end{matrix}\right\}$ order $\left\{\begin{matrix}N+1 \ (0)\\ N+2-s \ (N+2-s)\end{matrix}\right.$.

Proof. From AI (2,7 a) and AI (3,6 b) we get at once

$$\left.\begin{aligned} L_{a_{N+1}\,a_N\ldots a_1}^{\quad b_N\ldots b_1} &\equiv \Phi_{a_{N+1}\,a_N\ldots a_1}^{\quad b_N\ldots b_1} - \left\{ \Gamma_{a_{N+1}\,a_N}^{\quad b_N} \delta_{a_{N-1}\ldots a_1}^{b_{N-1}\ldots b_1} \right\} = \\ &= P_{a_{N+1}\ldots a_1}^{a'_{N+1}\ldots a'_1} P_{b'_N\ldots b'_1}^{b_N\ldots b_1} L_{a'_{N+1}\,a'_N\ldots a'_1}^{\quad b'_N\ldots b'_1} \end{aligned}\right\} \quad (1,3)$$

and this proves our theorem for $s=N$. Let now η^a be a privileged parameter system and $s<N$. Because of (1, 1) the elimination mentioned in the theorem leads to

$$\Phi_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1} = P_{a_{N+1}\ldots a_1}^{a'_{N+1}\ldots a'_1} P_{b'_s\ldots b'_1}^{b_s\ldots b_1} L_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\quad b'_s\ldots b'_1} \text{ at } P \quad (1,4)$$

where

$$\left.\begin{aligned} L_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\quad b'_s\ldots b'_1} &= \Phi_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\quad b'_s\ldots b'_1} + \sum_{s+1}^{N} {}_q\, \Phi_{a'_{N+1}\ldots a'_q\ldots a'_1}^{\quad d'_q\ldots d'_1} \Big[\left\{ \Omega_{d'_q\ldots d'_s}^{\quad b'_s} \delta_{d'_{s-1}\ldots d'_1}^{b'_{s-1}\ldots b'_1} \right\} \\ &+ \omega_{d'_q\ldots d'_s\ldots d'_1}^{\quad b'_s\ldots b'_1} \Big] - \left\{ \Omega_{a'_{N+1}\ldots a'_s}^{\quad b'_s} \delta_{a'_{s-1}\ldots a'_1}^{b'_{s-1}\ldots b'_1} \right\} - \omega_{a'_{N+1}\ldots a'_s\ldots a'_1}^{\quad b'_s\ldots b'_1}. \end{aligned}\right\} \quad (1,5)$$

Hence $L_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1}$ are components of an η-tensor defined in a privileged parameter system by (1, 2). The statement *b*) is obvious if we remember that $\Phi_{a_{N+1}\ldots a_s\ldots a_1}^{\quad b_s\ldots b_1}$ is of the $\left.\begin{matrix}\xi-\\ \eta-\end{matrix}\right\}$ order $\begin{matrix} N+1\ (0) \\ N+2-s\,(N+2-s) \end{matrix}$.

Let us now prove the following

Lemma (1, 2). *The* $\Xi_{a'_q\ldots a'_1}^{\quad c'}$ *which occur in* AI (4, 6) *have the following properties:*

a) *In a privileged parameter system their first derivatives are functions of the* $\Phi_{a'\ldots}^{b'\ldots}$ *and their first derivatives and*

b) *contain neither* $\Phi_{a\ldots}^{b\ldots}$ *nor their derivatives,*

c) *nor* $P_a^{a'}$, $P_{a'}^{a}$.

Proof. The derivative $P_{d'c'b'}^{\quad a}$ can be computed from AI(3, 6 *b*) and AI(4, 2). In doing so we have in a privileged parameter system because of the Lemma AI (4, 2)

$$P_{d'c'b'}^{\quad a} = P_{e'}^{a}\left[\Gamma_{d'a'}^{\quad e'}\Gamma_{c'b'}^{\quad a'} + \partial_{d'}\Gamma_{c'b'}^{\quad e'}\right] - P_{b'c'd'}^{b\,c\,d}\,\partial_d\Gamma_{cb}^{\ a} = P_{e'}^{a}\,\Omega_{d'c'b'}^{\quad e'} \text{ at } P,$$

or

$$\partial_d\Gamma_{cb}^{\ a} = P_{a'd\,c\,b}^{a\,d'c'b'}\left[\partial_{d'}\Gamma_{c'b'}^{\quad a'} + \Gamma_{d'e'}^{\quad a'}\Gamma_{c'b'}^{\quad e'} - \Omega_{d'c'b'}^{\quad a'}\right] \text{ at } P. \quad (1,6)$$

Hence $\partial_d\Gamma_{bc}^{\ a}$ expressed in a privileged parameter system does not

contain the $\Phi_{a\ldots}^{\;\;b\ldots}$ at all. On the other hand $\Xi_{d'c'b'}^{\quad\;\; a'}$ is a sum of product of at least one $\Gamma_{cb}^{\;\;a}$ and some $\Phi_{a'\ldots}^{\;\;b'\ldots}$. Hence according to our previous result $\partial_{e'}\Xi_{d'c'b'}^{\quad\;\; a'}$ expressed in a privileged parameter system contains neither the $\Phi_{a\ldots}^{\;\;b\ldots}$ nor their derivatives.

Furthermore we obtain from AI (4, 2) and AI (4, 6) in a privileged parameter system

$$\left.\begin{aligned} P_{e'd'c'b'}^{\qquad a} &= P_{a'}^{a}\,\Omega_{e'd'c'b'}^{\qquad a'} = \\ &= P_{a'}^{a}\left(\Gamma_{f'e'}^{\;\;a'}\,\Omega_{d'c'b'}^{\quad\;\; f'} + \partial_{e'}\,\Omega_{d'c'b'}^{\quad\;\; a'} + \partial_{e'}\,\Xi_{d'c'b'}^{\quad\;\; a'}\right) - P_{e'd'c'b'}^{e\,d\,c\,b'}\,\partial_e\,\Gamma_{dcb}^{\;\;\; a} \text{ at } P \end{aligned}\right\} \quad (1,7a)$$

or

$$\partial_e\,\Gamma_{dcb}^{\;\;\; a} = P_{e\,d\,c\,b\,a'}^{e'd'c'b'a}\left[-\Omega_{e'd'c'b'}^{\qquad a'} + \Gamma_{f'e'}^{\;\;a'}\,\Omega_{d'c'b'}^{\quad\;\; f'} + \partial_{e'}\left(\Omega_{d'c'b'}^{\quad\;\; a'} + \Xi_{d'c'b'}^{\quad\;\; a'}\right)\right] \text{ at } P. \quad (1,7b)$$

Hence $\partial_e\,\Gamma_{dcb}^{\;\;\; a}$ expressed in a privileged parameter system contains neither the $\Phi_{a\ldots}^{\;\;b\ldots}$ nor their derivatives and consequently $\partial_{f'}\,\Xi_{e'd'c'b'}^{\qquad a'}$ (which is a derivative of a sum of products of at least one of the $\Gamma_{ab}^{\;\;c}, \Gamma_{abc}^{\;\;\;d}$ and some $\Phi_{a'\ldots}^{\;\;b'\ldots}$ and eventually some $P_{a'}^{a}$, $P_{a}^{a'}$) expressed in a privileged parameter system contains neither $\Phi_{a\ldots}^{\;\;b\ldots}$ nor their derivatives. Continuing in this manner we may easily establish the statements a) and b) of our lemma. The proof of the statement c) is similar to that used in the proof of the lemma AI (4, 1).

If we put $\Omega_{c'b'}^{\;\;\;a'} \equiv \Gamma_{c'b'}^{\;\;\;a'}$ and $\Xi_{c'b'}^{\;\;\;a'} \equiv 0$ then the equation AI (4, 6) holds for $q = 2, 3, \ldots, N$. We use it in the following

Theorem (1, 2*a*). *The elimination of the* $P_{a'\ldots}^{\;\;\;b}$ *from* AI (4, 6) *for* $q = 3, \ldots, N$ *and from*

$$P_{b'_q\ldots b'_1}^{\qquad a} = \partial_{b'_q}\left[P_{a'}^{a}\,\Omega_{b'_{q-1}\ldots b'_1}^{\qquad\quad a'} + P_{a'}^{a}\,\Xi_{b'_{q-1}\ldots b'_1}^{\qquad\quad a'} - P_{b'_{q-1}\ldots b'_1}^{b_{q-1}\ldots b_1}\,\Gamma_{b_{q-1}\ldots b_1}^{\qquad\quad a}\right]^{1)} \quad (1,8)$$

$$q = 3, \ldots N$$

leads to a transformation law for an η*-tensor* $K_{b_q\ldots b_1}^{\qquad a}$, $(q = 3, \ldots, N)$ *which has the following properties:*

a) *its components are given in a privileged parameter system* η^a *by*

$$K_{b_q\ldots b_1}^{\qquad a} = -\,\partial_{b_q}\Gamma_{b_{q-1}\ldots b_1}^{\qquad\quad a} \text{ at } P; \qquad (q = 3, \ldots, N). \;.\;.\;\; (1,9)$$

b) *it is of the* $\left.\begin{matrix}\xi-\\ \eta-\end{matrix}\right\}$ *order* $\begin{matrix}N+2\,(1)\\ q\,(q)\end{matrix}$.

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1) The right hand member is the derivative of the right hand member of AI (4, 6) for $q - 1 = 2, \ldots, N - 1$.

Proof. In a privileged parameter system η^a the elimination leads to

$$\left.\begin{aligned} -\partial_{b_q} \Gamma_{b_{q-1}\ldots b_1}^{\;a} = P^{a\,b'_q\ldots b'_1}_{a'b_q\ldots b_1}\Big[\Omega_{b'_q\ldots b'_1}^{\;a'} - \Gamma_{f'b'_q}^{\;a'}\,\Omega_{b'_{q-1}\ldots b'}^{\;f'} - \\ -\partial_{b'_q}\Omega_{b'_{q-1}\ldots b'_1}^{\;a'} - \partial_{b'_q}\Xi_{b'_{q-1}\ldots b'_1}^{\;a'}\Big].\end{aligned}\right\} \quad (1, 10)$$

According to the Lemma (1, 2) the righthand member in brackets is a function of the $\Phi_{a'\ldots}^{\;b'\ldots}$ and their first derivatives only (while the lefthand member depends on $\Phi_{a\ldots}^{\;b\ldots}$ and their first derivatives only). If we denote it by $K_{b'_q\ldots b'_1}^{\;a'}$ we have from (1, 10)

$$-\partial_{b_q}\Gamma_{b_{q-1}\ldots b_1}^{\;a} = P^{b'_q\ldots b'_1 a}_{b_q\ldots b_1 a'}\,K_{b'_q\ldots b'_1}^{\;a'} \text{ at } P \quad . \; . \; . \quad (1, 11)$$

which proves the statement a). The statement b) is obvious (and follows at once from (1, 10)).

The set of η-tensors K can be completed:

Theorem (1, 2b). *The elimination of* $P_{b'_{N+1}\ldots b'}^{\;a}$ *and* $P_{b'_s\ldots b'_1}^{\;a}$, $s=2,\ldots,N$ *from* AI (2, 7c), AI (3, 6b), AI (4, 6) *and from*

$$P_{b'_{N+1}\ldots b'_1}^{\;a} = \partial_{b'_{N+1}}\Big[P^a_{a'}\,\Omega_{b'_N\ldots b'_1}^{\;a'} + P^a_{a'}\,\Xi_{b'_N\ldots b'_1}^{\;a'} - P^{b_N\ldots b_1}_{b'_N\ldots b'_1}\,\Gamma_{b_N\ldots b_1}^{\;a}\Big] \quad . \quad (1, 12)$$

leads to a transformation law for an η*-tensor* $K_{b_{N+1}\ldots b_1}^{\;a}$ *which has the following properties:*

a) *its components are given in a privileged parameter system* η^a *by*

$$K_{b_{N+1}\ldots b_1}^{\;a} = \Phi_{b_{N+1}\ldots b_1}^{\;a} - \partial_{b_{N+1}}\Gamma_{b_N\ldots b_1}^{\;a} \text{ at } P, \; . \quad . \quad . \quad (1, 13)$$

b) *it is of the* $\left.\begin{matrix}\xi- \\ \eta-\end{matrix}\right\}$ *order* $\begin{matrix}N+2\ (1) \\ N+1\ (N+1)\end{matrix}$.

The proof is similar to the previous one.

The tensors L and K occur in the *Frenet formulae* which we are going to establish.

2. *Frenet formulae.*

If $X_{\ldots a\ldots}^{\ldots\nu\ldots}$ is a $(\xi\eta)$-tensor with both kinds of indices then it is known [2]) that its covariant derivative is

$$\begin{aligned} D_b X_{\ldots a\ldots}^{\ldots\nu\ldots} &= \partial_b X_{\ldots a\ldots}^{\;\nu} + \ldots + \Gamma_{\mu\lambda}^{\nu} T_b^{\lambda} X_{\ldots a\ldots}^{\ldots\mu\ldots} + \ldots - \Gamma_{ab}^{\;c} X_{\ldots c\ldots}^{\ldots\nu\ldots} + \ldots = \\ &= \nabla_b X_{\ldots a\ldots}^{\ldots\nu\ldots} + \ldots - \Gamma_{ab}^{\;c} X_{\ldots c\ldots}^{\ldots\nu\ldots} + \ldots . \end{aligned}$$

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2) Cf. J. A. SCHOUTEN–D. J. STRUIK: Einführung in die neueren Methoden der Differentialgeometrie (Groningen, 1938).

Thus for instance

$$D_b T^{\nu}_{a} = T^{\nu}_{ba} - \Gamma^{c}_{ab} T^{\nu}_{c}, \quad D_c H^{\nu}_{ba} = \nabla_c H^{\nu}_{ba} - \Gamma^{d}_{bc} H^{\nu}_{da} - \Gamma^{d}_{ac} H^{\nu}_{bd} \; . \; . \qquad (2,1)$$

We use this notation in the following theorem about the $H^{\nu}_{a_x \ldots a_1}$ (cf. the theorem $AI(5,2)$):

Theorem (2, 1). *The* ($\xi\eta$)*-tensors* T^{ν}_{a}, $H^{\nu}_{a_x \ldots a_1}$, $(x = 2, \ldots, N)$ *satisfy the following equations (Frenet formulae):*

$$\left.\begin{array}{ll}
\text{a)} & D_{a_2} T^{\nu}_{a_1} = H^{\nu}_{a_2 a_1}, \\
\text{b)} & D_{a_3} H^{\nu}_{a_2 a_1} = H^{\nu}_{a_3 a_2 a_1} + K_{a_3 a_2 a_1}{}^{b} T^{\nu}_{b}, \\
\text{c)} & D_{a_{q+1}} H^{\nu}_{a_q \ldots a_1} = H^{\nu}_{a_{q+1} \ldots a_1} + \sum\limits_{2}^{q-1} {}_r H^{\nu}_{b_r \ldots b_1} \left\{ K_{a_{q+1} \ldots a_r}{}^{b_r} \delta^{b_{r-1} \ldots b_1}_{a_{r-1} \ldots a_1} \right\} + \\
& \qquad + K_{a_{q+1} \ldots a_1}{}^{b} T^{\nu}_{b} \;, \quad (q = 3, \ldots, N-1) \\
\text{d)} & D_{a_{N+1}} H^{\nu}_{a_N \ldots a_1} = L_{a_{N+1} a_N \ldots a_1}{}^{b_N \ldots b_1} H^{\nu}_{b_N \ldots b_1} + \\
& \quad + \sum\limits_{2}^{N-1} {}_r H^{\nu}_{b_r \ldots b_1} \left[\left\{ K_{a_{N+1} \ldots a_r}{}^{b_r} \delta^{b_{r-1} \ldots b_1}_{a_{r-1} \ldots a_1} \right\} + L_{a_{N+1} \ldots a_r \ldots a_1}{}^{b_r \ldots b_1} \right] + K_{a_{N+1} \ldots a_1}{}^{b} T^{\nu}_{b} \text{ [3]}.
\end{array}\right\} \quad (2,2)$$

Proof. The equation (2, 2*a*) is obvious (cf. A (5, 1*a*), AI (5, 6) and (2, 1)). In order to prove the remaining equations let us introduce a privileged parameter system η^a. Then we have for the γ which appears in the equation of the footnote 7) in AI

$$\partial_c \gamma = 0 \text{ at } P \text{ [4]}. \; . \; . \; . \; . \; . \; . \; . \qquad (2,3)$$

and moreover because of $AI(5,1)$ and AI (5, 6) and (2, 3)

$$\left.\begin{array}{l}
D_{a_{q+1}} H^{\nu}_{a_q \ldots a_1} = \nabla_{a_{q+1}} H^{\nu}_{a_q \ldots a_1} = \\
\qquad = T^{\nu}_{a_{q+1} \ldots a_1} - \sum\limits_{2}^{q-1} {}_r T^{\nu}_{b_r \ldots b_1} \left\{ \partial_{a_{q+1}} \Gamma_{a_q \ldots a_r}{}^{b_r} \delta^{b_{r-1} \ldots b_1}_{a_{r-1} \ldots a_1} \right\} - \\
\qquad\qquad - T^{\nu}_{b} \partial_{a_{q+1}} \Gamma_{a_q \ldots a_1}{}^{b} \text{ at } P \;, \quad q = 3 \ldots N-1.
\end{array}\right\} \quad (2,4)$$

From (1, 9) and (2, 4) we obtain at once (2, 2*c*) at a generator point P.

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[3]) If $N = 2$, then we have only (2, 2*a*) and (2, 2*d*), where $\sum\limits_{2}^{N-1} {}_r \equiv 0$.

[4]) γ is a sum of products of at least two of the Γ's (cf. the footnote AI [7])). On the other hand we have $\Gamma = 0$ at P in a privileged parameter system.

The equation (2, 2b) may be proved by a similar device. Finally we get in a privileged parameter system because of AI(2, 1)

$$D_{a_{N+1}} \overset{\nu}{H}_{a_N\dots a_1} = \nabla_{a_{N+1}} \overset{\nu}{H}_{a_N\dots a_1} =$$

$$= \Phi_{a_{N+1}a_N\dots a_1}^{\quad b_N\dots b_1} \overset{\nu}{T}_{b_N\dots b_1} - \sum_{2}^{N-1}{}_{r}\, \overset{\nu}{T}_{b_r\dots b_1}\Big[-\Phi_{a_{N+1}\dots a_r\dots a_1}^{\quad b_r\dots b_1} + \Big\{\partial_{a_{N+1}}\Gamma_{a_N\dots a_r}^{\ b_r}\,\delta_{a_{r-1}\dots a_1}^{b_{r-1}\dots b_1}\Big\}\Big]$$

$$- \overset{\nu}{T}_b\Big(\partial_{a_{N+1}}\Gamma_{a_N\dots a_1}^{\ b} - \Phi_{a_{N+1}\dots a_1}^{\ b}\Big) \text{ at } P$$

and this equations, together with (1, 2), (1, 13) and (1, 9), lead to (2, 2d) at the generator point P.

Corollary (2, 1). *A necessary and sufficient condition for*

$$\triangle_{a_q}\dots\triangle_{a_2}\overset{\nu}{T}_{a_1} = D_{a_q}\dots D_{a_2}\overset{\nu}{T}_{a_1}{}^{5)}, \qquad q=3,\dots,N \quad . \quad (2,5)$$

is

$$K_{a_q\dots a_1}^{\ b} = 0 \qquad q=3,\dots,N. \quad . \quad . \quad . \quad . \quad . \quad (2,6)$$

Proof. From (2. 6) and (2, 2) we have

$$D_{a_q}\overset{\nu}{H}_{a_{q-1}\dots a_1} = \overset{\nu}{H}_{a_q\dots a_1} = D_{a_q}\dots D_{a_2}\overset{\nu}{T}_{a_1}, \qquad q=3,\dots,N \quad (2,7)$$

and this equation is equivalent to (2, 5). From (2, 5) we have (2, 7) and because of (2, 2), the equation (2, 6) has to be satisfied.

Note I. The case $L=0$ will be treated in the next section.

Note II. If we are dealing with the maximal case with $m_N = n$, then the ξ-vectors $\overset{\nu}{T}_a$, $\overset{\nu}{H}_{a_x\dots a_1}$, $(x=2,\dots,N)$ are linearly independent and moreover there is only one set of ξ-vectors $H_\nu^{a_q\dots a_1}$, $q=1,\dots,N$ satysfying the conditions

$$\left.\begin{array}{c} \overset{\nu}{H}_{a_q\dots a_1} H_\nu^{b_q\dots b_1} = \delta_{a_q\dots(a_2a_1)}^{b_q\dots(b_2b_1)}, \quad \overset{\nu}{H}_{a_s\dots a_1} H_\nu^{b_q\dots b_1} = 0, \\ \Big(s, q=1,\dots,N,\ s\neq q,\ \overset{\nu}{H}_a \equiv \overset{\nu}{T}_a\Big). \end{array}\right\} \quad . \quad . \quad (2,8)$$

Hence we have from (2, 2)

$$\left.\begin{array}{ll} \text{a)} & K_{a_{q+1}\dots a_1}^{\ b} = H_\nu^{b} D_{a_{q+1}}\overset{\nu}{H}_{a_q\dots a_1}, \qquad q=2,\dots,N \\ \text{b)} & L_{a_{N+1}a_N\dots a_2a_1}^{\quad b_N\dots(b_2b_1)} = H_\nu^{b_N\dots b_1} D_{a_{N+1}}\overset{\nu}{H}_{a_N\dots a_1} \\ \text{c)} & L_{a_{N+1}\dots a_r\dots a_2a_1}^{\quad b_r\dots(b_2b_1)} = H_\nu^{b_r\dots b_1} D_{a_{N+1}}\overset{\nu}{H}_{a_N\dots a_1} - \Big\{K_{a_{N+1}\dots a_r}^{\ b_r}\,\delta_{a_{r-1}\dots a_1}^{b_{r-1}\dots b_1}\Big\} \\ & \qquad r=2,\dots,N-1. \end{array}\right\} (2,9)$$

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5) Cf. the theorem AI (5,1).

3. *The case $m=1$.*

In this section we shall deal with a curve in an A_n, $m=1$, $N=n$. All latin (covariant or contravariant) indices reduce to 1 and we shall in general drop this index as well as any group of $n+1$ latin (covariant or contravariant) indices, introducing the following simplifications

$$\eta,\ P_{q'},\ \Phi^{q},\ \Phi^{q'},\ \Gamma_{q},\ \Gamma_{q'},\ T^{\nu},\ H^{\nu}_{q},\ \frac{\delta}{\delta\eta},\ D_{\eta} \quad \ldots \quad (3,1)$$

for

$$\left.\begin{array}{l} \eta^{a},\ P^{\ \ \ \ \ \ \ b}_{a'_q\ldots a'_1},\ \Phi^{\ \ \ \ \ \ \ \ \ \ \ \ \ b_q\ldots b_1}_{a_{N+1}\ldots a_q\ldots a_1},\ \Phi^{\ \ \ \ \ \ \ \ \ \ \ \ \ b'_q\ldots b'_1}_{a'_{N+1}\ldots a'_q\ldots a'_1},\ \Gamma^{\ \ \ \ \ \ \ b}_{a_q\ldots a_1},\ \Gamma^{\ \ \ \ \ \ \ b'}_{a'_q\ldots a'_1} \\ \qquad T^{\nu}_{a},\ H^{\nu}_{a_q\ldots a_1},\ \nabla_{a},\ D_{a}. \qquad (q=2,\ldots,N). \end{array}\right\} \quad (3,2)$$

Moreover we write η' P for $\eta^{a'}$, $P^{a}_{a'}$.

Theorem (3, 1). *A parameter s (affine arc) may be always found for which*

$$\Gamma_2=0 \ \textit{along}\ A_1 \quad \ldots\ldots\ldots \quad (3,3)$$

and consequently

$$\frac{\delta}{\delta s}=D_s \ \textit{along}\ A_1 \quad \ldots\ldots\ldots \quad (3,4)$$

This parameter is given by

$$\left.\begin{array}{l} s=c_1\displaystyle\int e^{\int \Gamma_{2'}\,d\eta'}\,d\eta'+c_2, \\ (c_1,\ c_2\ \textit{const.},\ c_1\neq 0) \end{array}\right\} \quad \ldots\ldots \quad (3,5)$$

where η' is a generator parameter.

Proof. The equation AI (3, 6*b*) reduces for $m=1$ to

$$P_{2'}=P\,\Gamma_{2'}-(P)^2\,\Gamma_2.$$

Substituting in it $\eta\equiv s$ and (3, 3), we get a differential equation

$$\frac{d^2 s}{d\eta'^2}-\Gamma_{2'}\frac{ds}{d\eta'}=0 \quad \ldots\ldots\ldots \quad (3,6)$$

with the solution (3, 5). The equation (3, 4) is obvious.

Theorem (3, 2). *If $m=1$, then*

$$L^{\ \ \ \ \ \ \ \ \ \ \ \ \ b_q\ldots b_1}_{a_{N+1}\ldots a_q\ldots a_1}=0, \qquad q=2,\ldots,N \quad \ldots\ldots \quad (3,7)$$

Proof. The symbol $\left\{P^{\ \ \ \ \ \ \ \ b_N}_{a'_{N+1}a'_N}\,P^{b_{N-1}\ldots b_1}_{a'_{N-1}\ldots a'_1}\right\}$ used in AI (2, 7a) reduces

for $m=1$ to $(P)^{N-1} P_{2'} C_1$, where $C_1 \neq 0$ is a conveniently chosen integer. Hence AI (2, 7a) reduces for $m=1$ to

$$\Phi^N = (P)^{-1} \Phi^{N'} - C_1 (P)^{-2} P_{2'} \quad (3, 8a) \quad , \quad \text{or } P_{2'} = \frac{P}{C_1} (\Gamma^{N'} - P \Phi^N). \quad (3, 8b)$$

If we substitute for $P_{2'}$ from (3,8 b) in (3,8 a), we get the identity $0=0$. Hence (3,7) holds for $q=N$. The same device applied to AI (2,7 b) leads to (3,7) for $q=2, \ldots, N-1$.

Note. If we write K_s for $K_{a_s \ldots a_1}^{b}$, $s=3, \ldots, N+1$, then the symbol $\left\{ K_{a_{q+1} \ldots a_r}^{b_r} \delta_{a_{r-1} \ldots a_1}^{b_{r-1} \ldots b_1} \right\}$ used in (2,2 c) reduces to $C_{q-r+1} K_{q-r+2}$, where $C_{q-r+1} \neq 0$ is a conveniently chosen integer which does not depend on our curve:

Theorem (3,3). *The Frenet formulae* (2,2) *reduce for* $m=1$ *to*

$$\left.\begin{aligned} &\text{a)} \quad \frac{\delta}{\delta s} T^\nu = H_2^\nu, \qquad b) \quad \frac{\delta}{\delta s} H_2^\nu = H_3^\nu + K_3 T^\nu \\ &\frac{\delta}{\delta s} H_q^\nu = H_{q+1}^\nu + \sum_2^{q-1} {}_r\, C_{q-r+1} H_r^\nu K_{q-r+2} + K_{q+1} T^\nu \\ &\qquad\qquad\qquad (q=3, \ldots, N, \; H_{N+1}^\nu = 0). \end{aligned}\right\} \quad (3, 9)$$

The proof follows at once from (2,2). (3,4) and (3,7).

Theorem (3,4). *The functions* $K_q(s)$, $q=3, \ldots, N+1$ *define a curve whose Frenet formulae are* (3,8), *up to the initial conditions* ξ (P), $T^\nu(P)$, $H_x^\nu(P)$ $x=2, \ldots, N$.

The proof may be obtained by a classical device from (3,9).

Note. If $n=2$ and A_2 is a plane, then $K_3(s) = const.$ defines a conic section and in particular this conic section is a hyperbole, parabole or ellipse if $K_3 > 0$, $K_3 = 0$ or $K_3 < 0$.

4. *Internal contact invariants.*

Let

$$\text{a)} \quad \xi^\nu = \varphi^\nu(\eta^1, \ldots, \eta^m) \quad \text{and} \quad b) \quad \xi^\nu = \psi^\nu('\eta^1, \ldots, '\eta^m) \; . \quad (4, 1)$$

be the parametric equations of two *maximal* subspaces A_m and $'A_m$ which have a real point P in common. Let

$$\eta^a = H^a('\eta) \quad , \quad '\eta^a = 'H^a(\eta) \quad . \; . \; . \; . \; . \quad (4, 2)$$

be a one-to-one mapping of A_m onto $'A_m$, which preserves the point P and put $\psi^\nu('H) \equiv '\varphi^\nu(\eta)$:

Definition (4, 1). *We say that* A_m *and* $'A_m$ *have at* P *internal contact of order* r *if*

$$\frac{\partial^s \varphi^\nu}{\partial \eta^{a_s} \ldots \partial \eta^{a_1}} = \frac{\partial^s \, '\varphi^\nu}{\partial \eta^{a_s} \ldots \partial \eta^{a_1}} \text{ at } P. \quad \ldots \ldots \quad (4, 3)$$
$$s = 1, \ldots, r.$$

Note I. It may be easily shown that this definition is $(\xi\eta)$-invariant.

Note II. All objects of $'A_m$ will be designated by the same letters as the corresponding objects of A_m but for the primes. Throughout this section we shall suppose $N = 'N$.

Theorem (4, 1). *If* A_m *and* $'A_m$ *have internal contact of order* $r \leqq N$ *at* P, *then we have*

$$\overset{s}{E}_{m_s} = '\overset{s}{E}_{'m_s} \text{ at } P \text{ [6]}. \quad \ldots \ldots \quad (4, 4)$$
$$s = 1, \ldots, r.$$

Proof. The ξ-vector $T^\nu_{a_s \ldots a_1}$ is a function of $\dfrac{\partial^p \varphi^\nu}{\partial \eta^{a_p} \ldots \partial \eta^{a_1}}$, $p = 1, \ldots, s$

and of $\Gamma^\alpha_{\gamma\beta}(\varphi)$ and their $s-2$ derivatives. Hence according to the definition (4, 1) we have for the internal contact of order r at P

$$T^\nu_{a_p \ldots a_1} - 'T^\nu_{a_p \ldots a_1} \text{ at } P, \qquad p = 1, 2, \ldots, r \quad \ldots \quad (4, 5)$$

and consequently (4, 4) holds.

Theorem (4, 2). *If* A_m *and* $'A_m$ *have internal contact of order* $r = N + 1$ *at* P *then* (4, 4) *holds for* $s = 1, \ldots, N$ *and moreover*

$$a) \;\; \overset{q}{N}_{n_q} = '\overset{q}{N}_{'n_q} \text{ [7]}, \quad b) \;\; L_{a_N \ldots a_x \ldots a_1}^{\;\;b_x \ldots b_1} = 'L_{a_N \ldots a_x \ldots a_1}^{\;\;b_x \ldots b_1} \text{ at } P \;. \quad (4, 6)$$
$$q = 1, \ldots, N-1, \;\; x = 2, \ldots, N.$$

Proof. If $r = N + 1$ then (4, 5) holds for $p = 1, \ldots, N + 1$ and consequently

$$\sum_{1}^{N} {}_x \left(\Phi_{a_{N+1} \ldots a_x \ldots a_1}^{\;\;b_x \ldots b_1} - '\Phi_{a_{N+1} \ldots a_x \ldots a_1}^{\;\;b_x \ldots b_1} \right) T^\nu_{b_x \ldots b_1} = 0 \text{ at } P. \;. \quad (4, 7)$$

In our maximal case the ξ-vectors $T^\nu_{a_x \ldots a_1}$ are linearly independent and therefore (4, 7) is equivalent to

$$\Phi_{a_{N+1} \ldots a_x \ldots a_1}^{\;\;b_x \ldots b_1} = '\Phi_{a_{N+1} \ldots a_x \ldots a_1}^{\;\;b_x \ldots b_1} \text{ at } P, \qquad x = 1, \ldots, N. \;. \quad (4, 8a)$$

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[6] Cf. the first section in *AI*.

[7] Cf. the definition *AI* (5,1).

Hence

$$\Gamma_{a_q\ldots a_1}^{\;b} = {}'\Gamma_{a_q\ldots a_1}^{\;b} \text{ at } P, \qquad q=2,\ldots,N. \quad . \quad . \quad (4,8b)$$

The equations (4, 5) for $p=1,\ldots,N$ and (4, 8) lead at once to (4, 6 a, b). In order to prove the next theorem we need the following

Lemma (4, 1). *If A_m and $'A_m$ have internal contact of order $r=N+s$ $(s=2,3,\ldots)$ at P, then we have*

$$\frac{\partial^p}{\partial\eta^{c_p}\ldots\partial\eta^{c_1}}\Phi_{a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} = \frac{\partial^p}{\partial\eta^{c_p}\ldots\partial\eta^{c_1}}{}'\Phi_{a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} \text{ at } P. \quad (4,9)$$

$$p=1,2,\ldots,s-1, \quad x=1,\ldots,N.$$

Proof. We may put

$$T^{\nu}_{d_p\ldots d_1 a_{N+1}\ldots a_1} = \sum_1^N{}_x\, T^{\nu}_{b_x\ldots b_1}\Phi_{d_p\ldots d_1 a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} \quad . \quad (4,10)$$

with

$$\left.\begin{aligned}\Phi_{d_p\ldots d_1 a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} &= \frac{\partial^p}{\partial\eta^{d_p}\ldots\partial\eta^{d_1}}\Phi_{a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} + \\ &+ \psi_{d_p\ldots d_1 a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1},\end{aligned}\right\} \quad . . \quad (4,11)$$

where ψ depends on $\Phi_{a_{N+1}\ldots a_y\ldots a_1}^{\;b_y\ldots b_1}$, $(y=1,\ldots,N)$ and their first $p-1$ derivatives.

Similar equations hold for $'A_m$ and consequently we have [on account of (4, 5) for $r=N+s$]

$$\left.\begin{aligned}&\frac{\partial^p}{\partial\eta^{d_p}\ldots\partial\eta^{d_1}}\left(\Phi_{a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} - {}'\Phi_{a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1}\right) = \\ &{}'\psi_{d_p\ldots d_1 a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} - \psi_{d_p\ldots d_1 a_{N+1}\ldots a_x\ldots a_1}^{\;b_x\ldots b_1} \text{ at } P, \quad (p=1,\ldots,s-1).\end{aligned}\right\} \quad (4,12)$$

If $p=1$, this equation reduces to (4, 9) for $p=1$ by virtue of (4, 8a) (the ψ being a function of the $\Phi_{a_{N+1}\ldots a_y\ldots a_1}^{\;b_y\ldots b_1}$ only). If $p=2$, the equation (4, 12) reduces to (4, 9) for $p=2$ because of (4, 8a) and (4, 9) for $p=1$. Proceeding in this way the lemma can be easily proved.

Theorem (4, 3). *If A_m and $'A_m$ have internal contact of order $r=N+s$, $(s=2,3,\ldots)$ then besides the equations* (4, 4), (4, 6) *the following equations hold at P*:

$$\left.\begin{aligned}&a)\; K_{a_q\ldots a_1}^{\;b} = {}'K_{a_q\ldots a_1}^{\;b}, \quad b)\; D_{d_u}\ldots D_{d_1}K_{a_q\ldots a_1}^{\;b} = {}'D_{d_u}\ldots{}'D_{d_1}{}'K_{a_q\ldots a_1}^{\;b} \\ &c)\; D_{d_p}\ldots D_{d_1}L_{a_{N+1}\ldots a_v\ldots a_1}^{\;b_v\ldots b_1} = {}'D_{d_p}\ldots{}'D_{d_1}{}'L_{a_{N+1}\ldots a_v\ldots a_1}^{\;b_v\ldots b_1} \\ &d)\; D_{a_x}\ldots D_{a_2}T^{\nu}_{a_1} = {}'D_{a_x}\ldots{}'D_{a_2}{}'T^{\nu}_{a_1} \\ &\quad q=3,\ldots,N+1, \quad u=1,2,\ldots,s-2, \quad p=1,2,\ldots,s-1, \\ &\qquad\qquad v=2,\ldots,N, \quad x=2,\ldots,s+1.\end{aligned}\right\} (4,13)$$

Proof. $D_{d_u} \ldots D_{d_1} K_{a_q \ldots a_1}^{\;b}$ depends on the $\Phi_{a_{N+1} \ldots}^{\;b \ldots}$ and their first $u+1$ derivatives. A similar statement holds for $'D_{d_u} \ldots 'D_{d_1}$ $'K_{a_q \ldots a_1}^{\;b}$. Hence, if $r = N + s$, $(s = 2. 3, \ldots)$, then (4, 13 a, *b*) hold because of (4, 9). A similar device leads to (4, 13 *c*). Furthermore $D_{a_x} \ldots D_{a_2} T_{a_1}^{\nu}$ depends on $T_{a_y \ldots a_1}^{\nu}$, $y = 1, \ldots, x$, on $\Phi_{a_{N+1} \ldots}^{\;b \ldots}$ and their first $x - 2$ derivatives. A similar statement holds for $'D_{a_x} \ldots 'D_{a_2} 'T_{a_1}^{\nu}$. Hence, if $r = N + s$, $(s = 2, 3, \ldots)$, then (4, 13 *d*) holds because of (4, 9) and (4, 5).

Note. If $m = 1$, the equations (4, 13) reduce to

$$\text{a) } K_q = 'K_q, \quad b) \ \frac{d^u}{d s^u} K_q = \frac{d^u}{d' s^u} 'K_q, \quad d) \ \frac{\delta^x}{\delta s^x} T^\nu = \frac{'\delta^x}{'\delta' s^x} 'T^\nu \quad . \qquad (4, 14)$$

Errata in AI:

1) In the last equation of the footnote 5) read $P_{a'_x \ldots a'_2}^{\;a_2}$ instead of $P_{a'_{x+1} \ldots a'_2}^{\;a_2}$.

2) In the statement *a*) of Theorem (4, 1) omit the first "and".

3) In the statement *b*) of Theorem (4, 2) read (*q*) instead of (0).

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Mathematics. — *On problems analogous to those of* GOLDBACH *and* WARING. By W. VERDENIUS [1]). (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of May 28, 1949.)

GOLDBACH's conjecture that "every even number $\geqq 4$ is equal to the sum of two primes" has not been proved up to the present time, although, after the publication of the discoveries of HARDY and LITTLEWOOD [2]) in this domain, this problem has had the attention of many mathematicians.

VINOGRADOW [3]) proved in 1937 that every sufficiently large odd number is the sum of three primes. A few months later several mathematicians [4]) proved in a similar way that among the positive even number $\leqq N$, there are at most $c_1 N \log^{-\gamma_1} N$ numbers, which cannot be written as the sum of two primes. Here γ_1 is an arbitrary number and c_1 a suitable number depending only on γ_1.

VINOGRADOW's method is very fertile in dealing with similar problems. I have treated in my thesis some analogous problems.

In order to abbreviate the argument I make the following conventions:

When we say that there is a fixed number with a certain property we mean that it is possible to find a number, possessing that property, which only depends on the fixed numbers already mentioned before.

The numbers $c_1, \ldots, c_8$ denote suitably chosen fixed positive numbers and $\gamma_1, \ldots, \gamma_{14}$ denote arbitrary fixed positive numbers.

X is an integer $\geqq 3$, which is not fixed. We denote $\log X$ always by x and p denotes a prime number; $e(v) = e^{2\pi i v}$.

The notations $\alpha \ll \beta$ and $\beta \gg \alpha$ mean that $\beta > 0$ and that there exists a fixed positive number c_2, such that $|\alpha| \leqq c_2 \beta$ is valid.

In my applications I start from two theorems of VAN DER CORPUT [5]), who has generalised HARDY–LITTLEWOOD–VINOGRADOW's method. In order to show that the conditions, occurring in these two theorems, are satisfied, we use the following propositions, due to SIEGEL–WALFISZ and VINOGRADOW.

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[1]) Summary of the author's thesis: Over problemen analoog aan die van GOLDBACH en WARING, Amsterdam 1948.

[2]) HARDY, G. H. and J. E. LITTLEWOOD, Some problems of „Partitio Numerorum" III. On the expression of a number as a sum of primes. Acta Math. **44** (1923) p. 1—70.

[3]) VINOGRADOW, I., Representation of an odd number as a sum of three primes. C. R. Acad. Sci. U.R.S.S. **15** (1937), p. 291—294.

[4]) Cf. CORPUT, J. G. VAN DER, Sur l'hypothèse de GOLDBACH pour presque tous les nombres pairs. Acta Arithmetica **2** (1937), p. 266—290.

[5]) CORPUT, J. G. VAN DER, Propriétés additives, Acta Arithmetica **3** (1939), p. 180—234. (Théorèmes fondamentaux A et B.)

Theorem 1. *If q is a positive integer and k an integer prime to q, then for each γ_2*

$$\sum_{\substack{p \leq X \\ p \equiv k \,(\mathrm{mod}\ q)}} 1 - \frac{1}{\varphi(q)} \int_2^X \frac{du}{\log u} \ll X x^{-\gamma_2}.$$

Theorem 2. *If k is a fixed positive integer, d a fixed integer $\neq 0$; $\alpha_0, \ldots, \alpha_k$ real numbers; $f(y) = \alpha_0 y^k + \ldots + \alpha_k$, then to each γ_3 corresponds a c_3, such that*

$$\sum_{p \leq X} e(d f(p)) \ll X x^{-\gamma_3}$$

is true for each α_0 with the property that the closed interval $(\alpha_0 - X^{-k} x^{c_3}, \alpha_0 + X^{-k} x^{c_3})$ contains no irreductible fraction with denominator $\leqq x^{c_3}$.

By means of the above mentioned method the following theorems were already proved before.

Theorem 3: *Every sufficiently large positive integer $\equiv 5$ (mod. 24) is equal to the sum of the squares of five primes.*

Theorem 4: *The number of positive integers $\leqq X$ and $\equiv 3$ (mod. 24) and $\not\equiv 0$ (mod. 5), which cannot be written as the sum of the squares of three primes is $\ll X x^{-\gamma_4}$.*

Of both these propositions I have proved generalisations. For this purpose I consider the system

$$\left.\begin{array}{l} t = \psi_1 (y_{11}, \ldots, y_{1s_1}) + \ldots + \psi_n \ (y_{n1}, \ldots, y_{ns_n}), \\ A_{\nu\sigma} \leqq y_{\nu\sigma} \leqq B_{\nu\sigma} \ (\nu = 1, \ldots, n;\ \sigma = 1, \ldots, s_\nu), \\ y_{\nu\sigma} \text{ is prime } (\nu = 1, \ldots, n;\ \sigma = 1, \ldots, s'_\nu). \end{array}\right\} \quad \ldots \quad (1)$$

Here t, $A_{\nu\sigma}$, $B_{\nu\sigma}$ are integers, $n, s_1, \ldots, s_n$ denote fixed positive integers, $s'_1, \ldots, s'_n$ are fixed integers $\geqq 0$ respectively $\leqq s_1, \ldots, s_n$. The functions $\psi_1, \ldots, \psi_n$ are fixed quadratic polynomials, such that the homogeneous quadratic part of ψ_ν ($\nu = 1, \ldots, n$) possesses a discriminant $\neq 0$. I impose on the parallelepipeda Y_ν

$$A_{\nu\sigma} \leqq y_{\nu\sigma} \leqq B_{\nu\sigma} \quad (\sigma = 1, \ldots, s_\nu)$$

the condition that there exists a transformation of the form [6])

$$y_1 = z_1,\ y_\sigma = z_\sigma + a z_1 \quad (\sigma = 2, \ldots, s)$$

with fixed a and with the following properties:

1. The function ψ is transformed into a polynomial χ, such that the total degree of χ is equal to its degree with respect to z_1.
2. By this transformation Y is changed into a set Z, which entirely lies within the parallelepiped Z'

$$A'_\sigma \leqq z_\sigma \leqq B'_\sigma \quad (\sigma = 1, \ldots, s)$$

[6]) The index ν has been omitted.

and contains Z''

$$A''_\sigma \leqq z_\sigma \leqq B''_\sigma \quad (\sigma = 1, \ldots, s).$$

3. In the interior of Z' we have

$$\left|\frac{\partial \chi}{\partial z_1}\right| \gg X^{1-\frac{1}{4}} \text{ and } \frac{\partial^2 \chi}{\partial z_1^2}\frac{\partial \chi}{\partial z_1} \geqq 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

4. For the sets Z, Z' and Z'' we have

$$\left.\begin{array}{l} A_\sigma^{(\tau)} > 3 \quad (\sigma = 1, \ldots, s';\ \tau = 1, 2); \\ A_\sigma^{(\tau)} \ll X^{\frac{1}{4}},\ B_\sigma^{(\tau)} \ll X^{\frac{1}{4}},\ B_\sigma - A_\sigma \gg X^{\frac{1}{4}} \quad (\sigma = 1, \ldots, s;\ \tau = 1, 2). \end{array}\right\} \quad . \quad . \quad (3)$$

I have proved the following generalisations of the theorems 3 and 4:

Theorem 5: *If $n \geqq 3$, $s_1 \geqq 2$ and $s_2 \geqq 2$, then two constants P and Q can be found, such that the system (1) has a solution for sufficiently large X and for each integer t with $PX \leqq t \leqq QX$, provided that t satisfies a certain congruence with a fixed modulus. The number of these solutions increases indefinitely with X.*

If it is possible to find $2n$ constants K_ν and L_ν ($\nu = 1, \ldots, n$) such that

$$\min_{y \text{ in } Y_\nu} \psi_\nu(y) \leqq K_\nu X < L_\nu X \leqq \max_{y \text{ in } Y_\nu} \psi_\nu(y) \qquad (\nu = 1, \ldots, n). \quad . \quad (4)$$

and

$$\sum_{\nu=1}^{n} K_\nu < 0 < \sum_{\nu=1}^{n} L_\nu, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

then we may choose $P < 0$ and $Q > 0$.

Theorem 6: *Suppose $n = 2$, $s_1 = s'_1 = 2$, $s_2 = s'_2 = 1$ and let ψ_1 and ψ_2 be homogeneous forms of second degree. In that case two constants P and Q can be found, with the following property:*

Consider the integers t with $PX \leqq t \leqq QX$ and satisfying a certain congruence with a fixed modulus; then the number of these integers t, for which the system (1) has no solutions, is $\ll X x^{-\gamma_5}$.

If it is possible to find $2n$ constants K_ν and L_ν ($\nu = 1, \ldots, n$) with (4) and (5), then we may choose again $P < 0$ and $Q > 0$.

Because these theorems are only interesting if X is sufficiently large, there is a great difference in the assertions according to whether P and Q have the same or opposite sign. For in the first case t increases with X if it lies in the interval $PX \leqq t \leqq QX$ and in the second case t can be choosen fixed. That is why we pay special attention to the signs of P and Q. I also give a method to compute an upperbound for P and a greater lowerbound for Q.

These theorems 5 and 6 are deduced from others, in which the number of the solutions of system (1) is considered. Besides this problem I also consider a more general one. Instead of the integer t we introduce m

numbers $t_1, \ldots, t_m$, in which m is fixed; further mn fixed numbers $b_{\mu\nu}$ ($\mu = 1, \ldots, m$; $\nu = 1, \ldots, n$).

The case in which $s'_1 = \ldots = s'_n = 0$ we call *the strong case* and the case in which at least one of these numbers is > 0 we call *the weak case.*

We denote by $L(t) = L(t_1, \ldots, t_m)$ the number of integral solutions

$$(y_{11}, \ldots, y_{1s_1}, \ldots, y_{n1}, \ldots, y_{ns_n})$$

of the system

$$\begin{cases} \sum_{\nu=1}^{n} b_{\mu\nu}\, \psi_\nu\, (y_{\nu 1}, \ldots, y_{\nu s_\nu}) = t_\mu & (\mu = 1, \ldots, m), \\ y_{\nu\sigma} \text{ is prime} & (\nu = 1, \ldots, n;\ \sigma = 1, \ldots, s'_\nu), \\ A_{\nu\sigma} \leqq y_{\nu\sigma} \leqq B_{\nu\sigma} & (\nu = 1, \ldots, n;\ \sigma = 1, \ldots, s_\nu). \end{cases}$$

We introduce the functions $\varrho_1(v), \ldots, \varrho_n(v)$, defined by

$$\varrho_\nu(v) = \underset{|\psi_\nu(y_1, \ldots, y_{s_\nu}) - v| \leqq \frac{1}{2}}{\underset{Y_\nu}{\int \cdots \int}} \frac{dy_1 \ldots dy_{s_\nu}}{\log y_1 \ldots \log y_{s'_\nu}} \qquad (\nu = 1, \ldots, n);$$

we put for every lattice point $t = (t_1, \ldots, t_m)$

$$\Lambda(t) = \Sigma_1 \prod_{\nu=1}^{n} \varrho_\nu(v_\nu),$$

where Σ_1 is extended over all lattice points $(v_1, \ldots, v_n)$ satisfying

$$t_\mu = \sum_{\nu=1}^{n} b_{\mu\nu}\, v_\nu \quad (\mu = 1, \ldots, m).$$

Further we denote by b^{-1} the number of points $\tau = (\tau_1, \ldots, \tau_m)$ with coordinates $\geqq 0$ and < 1 such that the n numbers $\sum_{\mu=1}^{m} b_{\mu\nu}\, \tau_\mu$ ($\nu = 1, \ldots, n$) are integers.

Finally we put

$$Q(p, t) = \lim_{\alpha \to \infty} Q_\alpha(p, t),$$

in which $Q_\alpha(p, t)$ is equal to

$$p^{-\alpha \sum_{\nu=1}^{n} s_\nu} \left(1 - \frac{1}{p}\right)^{-\sum_{\nu=1}^{n} s'_\nu}$$

times the number of incongruent solutions $(h_{\nu\sigma})$ of the system

$$\begin{cases} \sum_{\nu=1}^{n} b_{\mu\nu}\, \psi_\nu(h_{\nu 1}, \ldots, h_{\nu s_\nu}) \equiv t_\mu \,(\text{mod.}\ p^\alpha) \quad (\mu = 1, \ldots, m), \\ \prod_{\nu=1}^{n} \prod_{\sigma=1}^{s'_\nu} h_{\nu\sigma} \not\equiv 0 \,(\text{mod.}\ p). \end{cases}$$

Under the above conditions I have proved the following theorems.

Theorem 7. *Suppose that the matrix* $(b_{\mu\nu})$ $(\mu = 1, \ldots, m;\ \nu = 1, \ldots, n)$ *contains no determinant of the* m^{th} *order different from zero and that at least one of the following two conditions is fulfilled:*

1. *The functions* $\psi_1, \ldots, \psi_n$ *possess an equal number of variables; I denote this number by s. I suppose* $n \geqq 4m + 1$ *if* $s = 1$ *and* $n \geqq 2m + 1$ *if* $s \geqq 2$.

2. *Let be* $m = 1$ *and* $n \geqq 3$; *if* $n = 3$ or 4, *we assume that at most one, resp. two of the functions* $\psi_1, \ldots, \psi_n$ *contain only one variable.*

In the strong case there exists a constant c_4, *such that*

$$L(t) - b\,\Lambda(t) \prod_p Q(p, t) \ll X^{\sum_{\nu=1}^{n} \frac{1}{2} s_\nu - m - c_4}$$

holds uniformly in $t = (t_1, \ldots, t_m)$.

In the weak case we have for every constant γ_6

$$L(t) - b\,\Lambda(t) \prod_p Q(p, t) \ll X^{\sum_{\nu=1}^{n} \frac{1}{2} s_\nu - m} x^{-\gamma_6}$$

uniformly in t.

The product $\prod_p Q(p, t)$ *converges absolutely and uniformly in t.*

Theorem 8. *Suppose that the matrix* $(b_{\mu\nu})$ $(\mu = 1, \ldots, m;\ \nu = 1, \ldots, n)$ *contains no determinant of the* m^{th} *order different from zero and that at least one of the following two conditions is fulfilled:*

1. *The functions* $\psi_1, \ldots, \psi_n$ *possess an equal number of variables; I denote this number by s. Moreover* $n \geqq 2m + 3$ *if* $s = 1$; $n \geqq m + 2$ *if* $s = 2$ *and* $n \geqq m + 1$ *if* $s \geqq 3$.

2. *Let be* $m = 1$; $s_1 + \ldots + s_n \geqq 5$ *and* $n \geqq 2$.

Suppose also that the non-negative functions $w_\mu(t_\mu)$ $(\mu = 1, \ldots, m)$ *are defined for every lattice point* $t = (t_1, \ldots, t_m)$ *and satisfy*

$$\sum_{h=-\infty}^{+\infty} |w_\mu(h+1) - w_\mu(h)| \ll X^{-1} \sum_{h=-\infty}^{+\infty} w_\mu(h) \quad (\mu = 1, \ldots, m),$$

where the series on the right hand side converges. We put

$$w(t) = \prod_{\mu=1}^{m} w_\mu(t_\mu).$$

In the strong case there exists a constant c_5 *such that*

$$\sum_t w(t) \left| L(t) - b\,\Lambda(t) \prod_p Q(p, t) \right| \ll X^{\sum_{\nu=1}^{n} \frac{1}{2} s_\nu - m - c_5} \sum_t w(t)$$

and in the weak case we have for every γ_7

$$\sum_t w(t) \left| L(t) - b\,\Lambda(t) \prod_p Q(p, t) \right| \ll X^{\sum_{\nu=1}^{n} \frac{1}{2} s_\nu - m} x^{-\gamma_7} \sum_t w(t);$$

the sum $\sum_t$ *is extended over all lattice points t.*

The product $\prod\limits_p Q(p,t)$ *converges absolutely and uniformly in* t.

Moreover I have found

Theorem 9. *Suppose the conditions of theorem 5 are satisfied. I consider the relation*

$$L(t) = (1 + \theta\,\gamma_8\,x^{-\gamma_9})\,A(t)\,\prod_p{}_1\,Q(p,t),$$

where $|\theta| \leqq 1$ *and* $\gamma_8, \gamma_9, \gamma_{10}$ *and* γ_{11} *denote arbitrary positive constants; the product* $\prod\limits_p{}_1$ *is extended over all primes*

$$p \leqq X^{6+2\gamma_9+\gamma_{10}+\gamma_{11}}.$$

Then the number of the integers t *with* $PX \leqq t \leqq QX$, *satisfying a certain congruence with a fixed modulus, such that the relation in question is not satisfied, is* $\ll X\,x^{-\gamma_{10}}$.

In order to obtain these results I have given a comprehensive theory of generalised GAUSSian sums. This theory has some resemblance to WEBER's method [7]), but the sums occurring in my theory are mostly extended over reduced residu classes, whereas those in WEBER's theory are extended over complete classes, so that my investigation is much more complicated.

I have also dealt with the case that the polynomials ψ_ν are not of the second degree, but of an arbitrarily fixed degree k_ν. Even in that case it is possible to apply VAN DER CORPUT's theorems, but the estimation of the singular series and the error term offers greater difficulties. In that case we impose on the parallelepipeda Y_ν the same conditions as above, but in the inequalities (2) and (3) the exponent $\frac{1}{2}$ must be replaced by k^{-1}.

In order to be able to apply VAN DER CORPUT's theorems successfully I must unfortunately insert a complicated condition, which I formulate as follows:

Condition A. We say that the ml integers $\beta_{\mu\lambda}$, the l polynomials $\chi_\lambda(y)$ of the degree k_λ and with s_λ variables, and the s_λ-dimensional sets Y'_λ of lattice points ($\mu = 1, \ldots, m$; $\lambda = 1, \ldots, l$) satisfy the condition A, if there are l numbers r_λ ($\lambda = 1, \ldots, l$) with the following properties:

1. If N'_λ represents the number of systems formed by $2r_\lambda$ lattice points $(y_1, \ldots, y_{r_\lambda}, z_1, \ldots, z_{r_\lambda})$, all of them lying in Y'_λ, with

$$\sum_{\varrho=0}^{r_\lambda} (\chi_\lambda(y_\varrho) - \chi_\lambda(z_\varrho)) = 0,$$

then in the strong case

$$N'_\lambda \ll X^{\frac{2s_\lambda r_\lambda}{k_\lambda} - 1 + \gamma_{12}}$$

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[7]) WEBER, H., Ueber mehrfache GAUSSische Summen. Journal f. d. r. u. a. Math. **74**, (1872), p. 14—56.

and in the weak case

$$N'_\lambda \ll X^{\frac{2s_\lambda r_\lambda}{k_\lambda}-1} x^{c_8}.$$

2. The matrix $(\beta_{\mu\lambda})$ can be divided into submatrices, without common columns, all of the rank m, such that $r_\nu = r_\omega$ and $\frac{s_\nu}{k_\nu} = \frac{s_\omega}{k_\omega}$ for every couple of positive integers ν and ω possessing the property that the νth and the ωth column belong to the same submatrix.

3. To every submatrix mentioned in 2 a positive integer h corresponds, such that $r_\lambda = h$ if this submatrix contains the λth column of the matrix $(\beta_{\mu\lambda})$. Suppose $m(h)$ to be the total number of the submatrices for which the corresponding number is h. Now we assume

$$\sum_{h=1}^{\infty} \frac{m(h)}{h} \geqq 1.$$

I have proved the following theorem:

Theorem 10. *Suppose that to every positive integer $\lambda \leqq m$ a positive integer $l \leqq n-1$ and a permutation $(\lambda, \lambda_1, \ldots, \lambda_{n-1})$ of the system $(1, \ldots, n)$ corresponds, such that the matrix $(b_{\mu\lambda_\nu})$, the polynomials $\psi_{\lambda_\nu}(y_{\lambda_\nu})$ and the parallelepipeda Y_{λ_ν} satisfy condition A, not only with $\mu = 1, \ldots, m$; $\lambda = 1, \ldots, l$, but also with $\mu = 1, \ldots, m$; $\lambda = l+1, \ldots, n-1$. Suppose*

$$\begin{vmatrix} b_{11} \ldots b_{1m} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{m1} \ldots b_{mm} \end{vmatrix} \neq 0.$$

Suppose also, that the constants K_ν and L_ν $(\nu = 1, \ldots, n)$ can be chosen such that

$$\min_{y \text{ in } Y_\nu} \psi_\nu(y) \leqq K_\nu X < L_\nu X \leqq \max_{y \text{ in } Y_\nu} \psi_\nu(y) \qquad (\nu = 1, \ldots, n)$$

and that the system

$$\sum_{\nu=1}^{n} b_{\mu\nu} u_\nu = 0 \qquad (\mu = 1, \ldots, m)$$

has a solution in real numbers $(u_1, \ldots, u_n)$ with

$$K_\nu < u_\nu < L_\nu \qquad (\nu = 1, \ldots, n).$$

Under these conditions $2m$ constants $P_\mu < 0 < Q_\mu$ $(\mu = 1, \ldots, m)$ can be found, with the following properties:

Consider the lattice points $t = (t_1, \ldots, t_m)$ lying in the parallelepiped

$$P_\mu X \leqq t_\mu \leqq Q_\mu X \qquad (\mu = 1, \ldots, m)$$

for which

$$\sum_{\nu=1}^{n} b_{\mu\nu} v_\nu = t_\mu \qquad (\mu = 1, \ldots, m)$$

has a solution in integers $(v_1, \ldots, v_n)$; *then we have for sufficiently large* X

$$\Lambda(t) \gg X^{\sum_{\nu=1}^{n} \frac{s_\nu}{k_\nu} - m} \; x^{-\sum_{\nu=1}^{n} s'_\nu},$$

uniformly in the considered lattice points t.

In the strong case there exists a constant c_7 *such that*

$$L(t) - b\,\Lambda(t) \prod_p Q(p,t) \ll X^{\sum_{\nu=1}^{n} \frac{s_\nu}{k_\nu} - m - c_7}$$

uniformly in the considered lattice points t *and in the weak case we have for every constant* γ_{13}

$$L(t) - b\,\Lambda(t) \prod_p Q(p,t) \ll X^{\sum_{\nu=1}^{n} \frac{s_\nu}{k_\nu} - m} \cdot x^{-\gamma_{13}},$$

also uniform in these t.

The product $\prod_p Q(p,t)$ *converges absolutely and uniformly in the considered systems* t.

Concerning the numbers r_λ occurring in "condition A", I have proved the following two theorems. The first is of special interest in the strong case and in problems analogous to those of WARING it can be very useful. The second refers to the weak case for quadratic polynomials.

Theorem 11 [8]. *If* $r \geqq 2^k - 1$, *then the number* N *of the systems formed by* $2r$ *lattice points* $(y_1, \ldots, y_r, z_1, \ldots, z_r)$, *all lying in* Y, *with*

$$\sum_{\varrho=1}^{r} (\psi(y_\varrho) - \psi(z_\varrho)) = 0 \quad \ldots \ldots \ldots \quad (6)$$

satisfies for each γ_{14} *the inequality*

$$N \ll X^{\frac{2sr}{k} - 1 + \gamma_{14}}.$$

Theorem 12 [8]. *Let* r *be a positive integer and* $\psi(y)$ *a quadratic polynomial with* s *variables. The number* N *of the systems formed by* $2r$ *lattice points* $(y_1, \ldots, y_r; z_1, \ldots, z_r)$, *all lying in* Y, *with* (6), *satisfies the inequality*

$$N \ll X^{sr-1} x^{c_8},$$

by suitably chosen c_8, *provided that in the case* $r = 1$ *the polynomial* ψ *is not of the form*

$$a_0 (a_1 y_1 + \ldots + a_s y_s + a_{s+1})^2 + a_{s+2},$$

with fixed $a_0, \ldots, a_{s+2}$.

This last condition cannot be omitted.

With the aid of theorem 10 I also proved, that in some general cases $L(t)$ is positive. The most general theorem I proved is

[8]) The index λ has been omitted.

Theorem 13. *Suppose the conditions of theorem* 10 *are satisfied.*

Take a lattice point $t = (t_1, \ldots, t_m)$, *considered in theorem* 10 *and satisfying moreover the following condition:*

To each prime p corresponds an odd integer $\alpha > 0$, *such that the system*

$$\left.\begin{array}{l} \sum_{\nu=1}^{n} b_{\mu\nu}\, \psi_\nu (y_{\nu 1}, \ldots, y_{\nu s_\nu}) \equiv t_\mu \ (\mathrm{mod.}\ p^\alpha) \qquad (\mu = 1, \ldots, m), \\ \prod_{\nu=1}^{n} \prod_{\sigma=1}^{s'_\nu} y_{\nu\sigma} \not\equiv 0 \ (\mathrm{mod.}\ p) \end{array}\right\} \quad . \quad (7)$$

possesses at least one solution in integers $(y_{\nu\sigma})$ *with the property, that the corresponding matrix*

$$\begin{pmatrix} b_{11}\, \psi_{11} \ldots b_{11}\, \psi_{1s_1} \ldots b_{1n}\, \psi_{n1} \ldots b_{1n}\, \psi_{ns_n} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ b_{m1}\, \psi_{11} \ldots b_{m1}\, \psi_{1s_1} \ldots b_{mn} \psi_{n1} \ldots b_{mn} \psi_{ns_n} \end{pmatrix}$$[9],

has the rank m and that its last elementary divisor modulo p is not larger than $p^{\frac{1}{2}(\alpha-1)}$.

For such a lattice point t we have

$$\lim_{X \to \infty} L(t) = \infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If we consider a set of lattice points t, which all satisfy these conditions with the same α, *then* (8) *holds uniformly in these t.*

In the case of quadratic polynomials ψ_ν we prove the following theorem, which is a special case of the preceeding one.

Theorem 14. *Suppose the conditions of theorem* 10 *are sastisfied. Let the polynomials* ψ_ν *be of second degree with discriminant* D_ν *and let* $\triangle_\nu$ *be the discriminant of the homogeneous quadratic part of* ψ_ν $(\nu = 1, \ldots, n)$.

Take a lattice point $t = (t_1, \ldots, t_m)$, *considered in theorem* 10 *and satisfying moreover the following two conditions:*

1. *The system* (7) *has a solution in integers* $(y_{\nu\sigma})$, *for each power* p^α *of each prime p.*
2. *Let* T_μ *for* $\mu = 1, \ldots, m$ *be equal to*

$$2^{s_1 + \ldots + s_n} \left(t_\mu \prod_{\nu=1}^{n} \triangle_\nu - \sum_{\nu=1}^{n} b_{\mu\nu} \triangle_1 \ldots \triangle_{\nu-1} D_\nu \triangle_{\nu+1} \ldots \triangle_n \right),$$

and suppose that it is not possible to find $m-1$ *columns of the matrix* $(b_{\mu\nu})$ *such, that the column formed by* $T_1, \ldots, T_m$ *depends linearly on these columns.*

For such a lattice point t (8) *is satisfied.*

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[9]) Here $\psi_{\nu\sigma} = \dfrac{\partial}{\partial y_{\nu\sigma}} \psi_\nu (y_{\nu 1}, \ldots, y_{\nu s_\nu})$.

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Mathematics. — *On the impossibility of a just distribution.* By T. VAN AARDENNE–EHRENFEST. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 25, 1949.)

Let $c_1, c_2, \dots$ be an arbitrary infinite sequence of real numbers ("points") all belonging to the interval $0 \leqq t < 1$. Let $A_n(\alpha)$ denote the number of points belonging to the finite sequence $c_1, c_2, \dots c_n$ and lying in the interval α. Put

$$F_n(\alpha) = A_n(\alpha) - |\alpha|\, n,$$

where $|\alpha|$ is the length of α and let F_n be the least upper bound of $|F_n(\alpha)|$ where α runs through all subintervals of the interval $0 \leqq t < 1$.

We shall prove that

$$\operatorname*{Lim.\,sup.}_{n \to \infty} \left\{ F_n : \frac{\log \log n}{\log \log \log n} \right\} \geqq \tfrac{1}{2} \quad \dots \dots \quad \text{(I)}$$

From (I) it follows that a "just" distribution does not exist [1]). (A distribution is said to be "just" if F_n is bounded).

Remarks. 1. It does *not* follow that there does not exist a distribution such that $|F_n(\alpha)|$ is bounded for every subinterval α of the interval $0 \leqq t < 1$ by a number depending on α.

2. As far as I know for all special infinite sequences, for which F_n has been investigated, it has been found that

$$\operatorname*{Lim.\,sup.}_{n \to \infty} \{ F_n : \log n \} > 0,$$

which is much stronger than (I).

We consider a positive integer x and two intervals $a \leqq t < b$, $c \leqq t < d$ of equal length $b - a = d - c$ which have no points in common. We call these intervals ι and ι'. We put

$$v = (b - a) : 20^{10x!}.$$

In what follows Greek letters will always denote subintervals

$$a + kv \leqq t < a + lv$$

or

$$c + kv \leqq t < c + lv$$

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[1]) Cf. T. VAN AARDENNE–EHRENFEST, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **48**, 266 (1945). The proof given there is rather complicated and can be slightly simplified as is shown here. The impossibility of a just distribution has been conjectured by J. G. VAN DER CORPUT, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **38**, 816 (1935).

of ι and ι'. Here k and l are integers such that

$$0 \leqq k < l \leqq 20^{10x!}.$$

By $|\alpha|$ we denote the length of α. Further $\alpha \subset \beta$ means that α is a subinterval of β.

Moreover we consider a finite sequence of points $a_1, a_2, \ldots a_N$. If k, l are integers such that $0 \leq k < l \leq N$, we denote by $A_{kl}(\alpha)$ the number of points of α belonging to the sequence $a_{k+1}, a_{k+2}, \ldots a_l$.

We suppose that

$$A_{0N}(\iota) + A_{0N}(\iota') \geqq 20^{10x!}. \quad \ldots \ldots \quad (1)$$

Theorem. *There exists an index n $(n \leq N)$ and a pair σ, τ such that*

$$A_{0n}(\sigma) \geqq A_{0n}(\tau) + x, \quad |\tau| = |\sigma| + v \quad \ldots \quad (\mathrm{II})$$

and either $\sigma \subset \iota$, $\tau \subset \iota'$, or $\sigma \subset \iota'$, $\tau \subset \iota$.

If we take $a = 0$, $b = c = \frac{1}{2}$, $d = 1$, $N = 20^{10x!}$ and apply this theorem to the sequence $a_1 = c_1$, $a_2 = c_2$, $\ldots a_N = c_N$, it follows from (II) that either $|F_n(\sigma)| \geq \dfrac{x}{2}$ or $|F_n(\tau)| \geq \dfrac{x}{2}$. Since $n \leq 20^{10x!}$, we obtain (I).

Proof of the theorem. The theorem is obvious if $x = 1$. Therefore we suppose that $x \geq 2$ and that the theorem has already been established for $x - 1$ instead of x. Without loss of generality we suppose that

$$|\iota| = |\iota'| = 20^{10x!} \quad \ldots \ldots \ldots \quad (2)$$

and hence

$$v = 1 \quad \ldots \ldots \ldots \ldots \quad (3)$$

Now suppose that the theorem is *not* true for the given value of x and for the given sequence $a_1, a_2, \ldots a_N$. It follows that

$$A_{0n}(\xi) \leqq A_{0n}(\eta) + x - 1 \quad \text{for} \quad n = 1, 2, \ldots N \quad \ldots \quad (4)$$

whenever

$$|\eta| = |\xi| + 1$$

and either $\xi \subset \iota$, $\eta \subset \iota'$, or $\xi \subset \iota'$, $\eta \subset \iota$.

We deduce from (4)

$$A_{0n}(\xi) \leqq A_{0n}(\eta) + 2(x-1) \quad \text{for} \quad n = 1, \ldots N, \quad \ldots \quad (5)$$

whenever

$$|\eta| \geqq |\xi| + 2.$$

From (1) follows the existence of an index M $(1 < M < N)$ such that

$$A_{0M}(\iota) + A_{0M}(\iota') = 8x \cdot 20^{(x-1)(10(x-1)!+1)}. \quad \ldots \ldots \quad (6)$$

Now we shall prove two properties of the sequence $a_1, a_2, \ldots a_N$.

Property 1. *If*

$$|\alpha| = 20^{10(x-1)!-x+1},$$

then

$$A_{0M}(\alpha) \geqq 1\,, \qquad \qquad (7)$$

$$A_{0N}(\alpha) \geqq 10x+1, \qquad \qquad (8)$$

$$A_{0M}(\alpha) \leqq 10x\,, \qquad \qquad (9)$$

$$A_{MN}(\alpha) \geqq 1. \qquad \qquad (10)$$

Proof. If (7) were not true, then it would follow from (5) that $A_{0M}(\xi) \leq 2(x-1)$ whenever $|\xi| = \frac{1}{2}\cdot 20^{10(x-1)!-x+1}$. Hence, by (2),

$$A_{0M}(\iota)+A_{0M}(\iota') \leqq 2(x-1)\cdot 2.20^{10x!}\cdot 2.20^{-10(x-1)!+x-1} < 8x\cdot 20^{(x-1)(10(x-1)!+1)},$$

which contradicts (6).

If (8) were not true, then it would follow in the same way that

$$A_{0N}(\iota)+A_{0N}(\iota') \leqq \{2(x-1)+10x\}\cdot 2.20^{10x!}\cdot 2.20^{-10(x-1)!+x-1} < 20^{10x!},$$

which contradicts (1).

If (9) were not true then it would follow from (5) that

$$A_{0M}(\eta) > 10x-2(x-1) > 8x$$

whenever $|\eta| = 2.\,20^{10(x-1)!-x+1}$ and hence

$$A_{0M}(\iota)+A_{0M}(\iota') > 8x\cdot 2.20^{10x!}\cdot \tfrac{1}{2}\cdot 20^{-10(x-1)!+x-1} = 8x\cdot 20^{(x-1)(10(x-1)!+1)},$$

which contradicts (6).

(10) follows from (8) and (9).

Property 2. *If* $\omega \subset \iota$, $\omega' \subset \iota'$ *and*

$$|\omega| = |\omega'| = 20^{10k(x-1)!-x+k},$$

where k is a positive integer $\leq x$, then there exists a pair λ, μ such that

$$A_{0M}(\lambda) \geqq A_{0M}(\mu)+k\,, \quad |\mu| = |\lambda|+2(x-k)+1$$

and either $\lambda \subset \omega$, $\mu \subset \omega'$, or $\lambda \subset \omega'$, $\mu \subset \omega$.

In the special case that ω is identical with ι and ω' is identical with ι' it follows from this property (with $k = x$, cf. (2)) that there exists a pair σ, τ satisfying (II) with $n = M$ (cf. (3)). This proves the theorem.

Proof of property 2. First we consider the case $k = 1$. By (7) (property 1) we have

$$A_{0M}(\omega) \geqq 1$$

and hence there exists a λ such that $\lambda \subset \omega$, $|\lambda| = 1$ and $A_{0M}(\lambda) \geq 1$. On the other hand we have by (9)

$$A_{0M}(\omega') \leqq 10x$$

and therefore there exists a μ ($\mu \subset \omega'$) of the length $|\mu| = 2x$ which contains no point of the sequence $a_1, a_2, \dots a_M$. So we have

$$A_{0M}(\lambda) \geqq A_{0M}(\mu)+1\,, \quad |\mu| = |\lambda|+2(x-1)+1\,,$$

which proves property 2 in the case $k = 1$.

Therefore we suppose in what follows that $k \geqq 2$ and that property 2 has already been established for $k-1$ instead of k.

In order to find a pair λ, μ which has the required properties, we make the following construction.

1. We define φ and φ' as follows: $\varphi \subset \omega$, $\varphi' \subset \omega'$; φ has the distance

$$7.20^{10k(x-1)!-x+k-1}$$

from both endpoints of ω; φ' has the same distance from both endpoints of ω'. This involves that

$$|\varphi| = |\varphi'| = 6.20^{10k(x-1)!-x+k-1}.$$

Hence from (10) (property 1) it follows that

$$A_{MN}(\varphi) + A_{MN}(\varphi') > 20^{10(x-1)!}. \quad \ldots \ldots \quad (11)$$

Since we have supposed that the theorem is true for $x-1$ instead of x, we can apply it, by (11), to the sequence $a_{M+1}, a_{M+2}, \ldots a_N$ and to φ, φ' instead of ι, ι' with

$$v = 6.20^{10(k-1)(x-1)!-x+k-1}.$$

It results that there exists a pair σ, τ and an index n_0 $(M < n_0 \leq N)$ such that

$$A_{Mn_0}(\sigma) \geqq A_{Mn_0}(\tau) + (x-1), \quad \ldots \ldots \quad (12)$$

$$|\tau| = |\sigma| + 6.20^{10(k-1)(x-1)!-x+k-1} \quad \ldots \ldots \quad (13)$$

and either $\sigma \subset \varphi$, $\tau \subset \varphi'$ or $\sigma \subset \varphi'$, $\tau \subset \varphi$.

One of the intervals φ, φ' contains σ; we denote this interval by φ_σ; let φ_τ be the other of the intervals φ, φ'.

One of the intervals ω, ω' contains φ_σ; we denote this interval by ω_σ; let ω_τ be the other of the intervals ω, ω'. We have

$$\sigma \subset \varphi_\sigma \subset \omega_\sigma, \quad \tau \subset \varphi_\tau \subset \omega_\tau.$$

2. We define $\overline{\omega}_\sigma$ and $\overline{\omega}_\tau$ in the following way: $\overline{\omega}_\sigma$ is situated to the right of σ and is adjacent to it; $\overline{\omega}_\tau$ has the same right endpoint as τ;

$$|\overline{\omega}_\sigma| = |\overline{\omega}_\tau| = 20^{10(k-1)(x-1)!-x+k-1}.$$

Since the distance from φ_σ to the right endpoint of ω_σ is sufficiently large, we have $\overline{\omega}_\sigma \subset \omega_\sigma$.

3. We define δ in the following way: δ has the same left endpoint as τ,

$$|\delta| = 20^{10(x-1)!-x+1}.$$

From (13) follows: $\delta \subset \tau$, $\overline{\omega}_\tau \subset \tau$ and

$$\text{distance from } \delta \text{ to } \overline{\omega}_\tau > |\sigma| + |\overline{\omega}_\sigma|. \quad \ldots \ldots \quad (14)$$

From (7) (property 1) it results that $A_{0M}(\delta) \geqq 1$. Hence there exists an ε such that

$$\varepsilon \subset \delta, \quad |\varepsilon| = 1, \quad A_{0M}(\varepsilon) \geqq 1. \quad \ldots \ldots \quad (15)$$

4. Since we have supposed that property 2 is true with $k-1$ instead of k, there exists a pair $\bar{\lambda}, \bar{\mu}$ such that

$$A_{0M}(\bar{\lambda}) \geqslant A_{0M}(\bar{\mu}) + k-1, \quad |\bar{\mu}| = |\bar{\lambda}| + 2(x-k+1)+1 \quad . \quad (16)$$

and either $\bar{\lambda} \subset \bar{\omega}_\sigma$, $\bar{\mu} \subset \bar{\omega}_\tau$ or $\bar{\lambda} \subset \bar{\omega}_\tau$, $\bar{\mu} \subset \bar{\omega}_\sigma$.

We consider both cases separately.

First case. $\bar{\lambda} \subset \bar{\omega}_\sigma$, $\bar{\mu} \subset \bar{\omega}_\tau$. Let λ be the interval consisting of ε and of all points between ε and $\bar{\mu}$. We have $\lambda \subset \tau$ and hence

$$\lambda \subset \omega_\tau. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

We define μ as follows: μ is situated to the left of $\bar{\lambda}$ and is adjacent to it,

$$|\mu| = |\lambda| + 2(x-k)+1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Since the distance of φ_σ to the left endpoint of ω_σ is sufficiently large, we have

$$\mu \subset \omega_\sigma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Let ξ consist of μ and $\bar{\lambda}$. Since $|\mu|$ is sufficiently large (cf. (18) and (14)), we have $\sigma \subset \mu$, and hence

$$\sigma \subset \xi.$$

Let η be the interval consisting of $\bar{\mu}$ and of all points between ε and $\bar{\mu}$. We have

$$|\eta| = |\xi| + 1$$

and either $\xi \subset \iota$, $\eta \subset \iota'$, or $\xi \subset \iota'$, $\eta \subset \iota$.

Therefore, by (4),

$$A_{0n_0}(\xi) \leqslant A_{0n_0}(\eta) + x-1,$$

which can be written as

$$A_{0M}(\xi) + A_{Mn_0}(\xi) \leqslant A_{0M}(\eta) + A_{Mn_0}(\eta) + x-1.$$

Since $\sigma \subset \xi$ and $\eta \subset \tau$, we have by (12)

$$A_{Mn_0}(\xi) \geqslant A_{Mn_0}(\eta) + x-1.$$

Hence

$$A_{0M}(\xi) \leqslant A_{0M}(\eta),$$

which can be written as

$$A_{0M}(\mu) + A_{0M}(\bar{\lambda}) \leqslant A_{0M}(\lambda) + A_{0M}(\bar{\mu}) - A_{0M}(\varepsilon). \quad . \quad . \quad . \quad (20)$$

From (20), (16) and (15) it follows that

$$A_{0M}(\lambda) \geqslant A_{0M}(\mu) + k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

(17), (18), (19) and (21) show that the pair λ, μ has the required properties.

Second case. $\bar{\lambda} \subset \bar{\omega}_\tau$, $\bar{\mu} \subset \bar{\omega}_\tau$. Let in this case λ consist of ε, $\bar{\lambda}$ and of all points between ε and $\bar{\lambda}$. We define μ as follows: μ has the same right endpoint as $\bar{\mu}$ and

$$|\mu| = |\lambda| + 2(x-k) + 1.$$

Since the distance of φ_σ from the left endpoint of ω_σ is sufficiently large, we have again

$$\mu \subset \omega_\sigma.$$

By (14) we have again $|\mu| > |\lambda| > |\sigma| + |\bar{\omega}_\sigma|$ and hence $\sigma \subset \mu$. This means that $\sigma \subset \xi$, if ξ is the part of μ not belonging to $\bar{\mu}$ (for σ and $\bar{\mu}$ have no points in common). Let η consist of all points between ε and $\bar{\lambda}$. We have $\eta \subset \tau$ and

$$|\eta| = |\xi| + 1.$$

In the same way as in the first case we obtain

$$A_{0M}(\xi) \leqslant A_{0M}(\eta),$$

which in this case can be written as

$$A_{0M}(\mu) - A_{0M}(\bar{\mu}) \leqslant A_{0M}(\lambda) - A_{0M}(\bar{\lambda}) - A_{0M}(\varepsilon).$$

This is again the inequality (20) and leads to the same result as in the first case. So we have found in both cases a pair λ, μ which has the required properties.

Mathematics. — *On power series with integral coefficients.* I. By C. G. LEKKERKERKER. (Communicated by Prof. J. G. VAN DER CORPUT.)

(Communicated at the meeting of June 25, 1949.)

Introduction.

The starting-point of this paper is a result of H. GRAETZER. He constructs a power series with integral coefficients, which converges in the unit-circle and vanishes at an arbitrarily chosen point in the interior of the unit-circle [1]). In this paper I shall give generalizations of this result in various directions.

§ 1 contains the fundamental idea of this investigation. Here I prove the following theorem:

Let P and ω be real numbers with $0 < |p| < 1$, then there exists a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with *bounded* integral coefficients a_n, such that $f(p) = \omega$ (the series being convergent in the interior of the unit-circle).

Moreover I show that the coefficients can be chosen in such a way, that

$$|a_n| < \frac{1}{2|p|} + \frac{1}{2} \textit{ for } n = 1, 2, 3, \ldots.$$

In § 2 it is shown by methods from the theory of measure, that the last inequality is very sharp.

In § 3 a more general problem is treated, analogous to that of LAGRANGE in the theory of interpolation. Here I do not construct a *polynomial* in x, taking prescribed values for given values of x — as in the theory of interpolation — but I consider a *power series with bounded integral coefficients*, which has the same property. For reasons of convergence the given values of x must belong to the interval $-1 < x < 1$. For the solution of this problem it appeared to be convenient to introduce certain vectors and matrices.

In § 5 the method of § 3 is generalized; here I consider the analogue of the problem of HERMITE in the theory of interpolation. Moreover the problem is treated now in the complex domain.

In § 4 I evaluate a generalized determinant of VAN DER MONDE, which occurs in § 5.

In § 6 the notion "set of integers" is replaced by a more general one. For the proofs of the theorems in §§ 1, 3 and 5 we actually use only one property of the set of integers, from which the coefficients of the series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

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[1]) H. GRAETZER, Note on Power Series, J. London Math. Soc. 22, 90—92 (1947).

are taken, i.e.: every closed interval of unit length contains at least one integer. It therefore is natural to introduce in this theory a set A of real numbers with the property that there exists a positive number g, such that every closed interval of length g contains at least one number of A.

Moreover in the theorem of § 6 we do not introduce only one set A, but an infinite sequence of such sets $A_1, A_2, A_3, \dots$ with corresponding numbers $g_1, g_2, g_3, \dots$ and we require that in the series

$$f(x) = a_1 x + a_2 x^2 + \dots$$

a_1 belongs to A_1, a_2 to A_2, etc. The exact formulation of the theorem is too long to be reproduced here.

§ 1. *Let p and ω be real numbers with $0 < |p| < 1$. Then there exists a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with bounded integral coefficients a_n, such that $f(p) = \omega$.*

It is even possible to choose the integers a_n in such a way, that

$$|a_n| < \frac{1}{2|p|} + \frac{1}{2} \text{ for } n \geqq 1.$$

Proof. Without loss of generality we may suppose $|\omega| \leq \frac{1}{2}$. The coefficients in the series $\sum_{n=0}^{\infty} a_n x^n$ are determined in the following manner. We put $a_0 = 0$. Then we choose the integer a_1, so that

$$\frac{\omega}{p} = a_1 + \omega_1 \text{ with } |\omega_1| \leqq \frac{1}{2},$$

hence

$$|a_1| \leqq \left|\frac{\omega}{p}\right| + \frac{1}{2}.$$

In the last relation equality can occur only if $\frac{\omega}{p}$ is an integer $+\frac{1}{2}$; but then we can take for a_1 either of two consecutive integers; choosing the smallest absolute value we find

$$|a_1| < \left|\frac{\omega}{p}\right| + \frac{1}{2} \leqq \frac{1}{2|p|} + \frac{1}{2}.$$

We have

$$\omega = a_0 + a_1 p + \omega_1 p.$$

Now we define the integers $a_1, a_2, \dots$ by induction. Suppose that $a_1, a_2, \dots, a_n$ $(n \geq 1)$ already are determined in such a way, that

$$|a_m| < \frac{1}{2|p|} + \frac{1}{2} \qquad (m = 1, 2, \dots, n)$$

and

$$|\omega_m| \leqq \frac{1}{2} \qquad (m = 1, 2, \dots, n).$$

where ω_m is given by the relation

$$\omega = a_0 + a_1 p + \ldots + a_m p^m + \omega_m p^m \quad \ldots \ldots \quad (1)$$

Then we choose the integer a_{n+1}, so that

$$\frac{\omega_n}{p} = a_{n+1} + \omega_{n+1}$$

with

$$|\omega_{n+1}| \leqq \frac{1}{2} \text{ and } |a_{n+1}| < \frac{1}{2|p|} + \frac{1}{2}.$$

It follows

$$\begin{aligned}\omega &= a_0 + a_1 p + \ldots + a_n p^n + \omega_n p^n \\ &= a_0 + a_1 p + \ldots + a_n p^n + a_{n+1} p^{n+1} + \omega_{n+1} p^{n+1}.\end{aligned}$$

If the integers $a_0, a_1, a_2, \ldots$ are determined in this manner, then the relation (1) is true for any $m = 1, 2, \ldots$. On account of $|\omega_m| \leq \frac{1}{2}$ and $|p| < 1$ it follows

$$\omega = \sum_{n=0}^{\infty} a_n p^n \quad \ldots \ldots \ldots \ldots \quad (2)$$

N o t e. If $0 < p < 1$, then the above method can be varied as follows. We can suppose $0 \leq \omega < 1$. We use the conditions

$$0 \leqq a_n < \frac{1}{p}, \; 0 \leqq \omega_n < 1 \qquad (n \geqq 1).$$

It is possible to determine successively integers a_n and real numbers ω_n, given by the relation (1), such that these conditions are satisfied. As well as before we have $\sum_{n=0}^{\infty} a_n p^n = \omega$.

This last method is a generalization of the representation of a positive real number ω as a decimal in the scale of k. For if $\frac{1}{p}$ is an integer k, $f(p)$ has the form

$$f(p) = a_0 + a_1 \cdot \frac{1}{k} + a_2 \cdot \frac{1}{k^2} + \ldots + a_n \cdot \frac{1}{k^n} + \ldots,$$

where a_n is one of the integers $0, 1, 2, \ldots, k-1$ $(n \geq 1)$.

§ 2. The question arises, if it is possible to replace the inequality

$$|a_n| < \frac{1}{2|p|} + \frac{1}{2}$$

by a sharper one. I shall prove now the following theorem:

Let p be a real number with $0 < |p| < 1$. All numbers, which can be

represented by a power series $\sum_{n=0}^{\infty} a_n p^n$ *with integral coefficients* a_n, *so that*

$$|a_n| < \frac{1}{2|p|} - \frac{1}{2} \text{ for } n \geqq 1,$$

form a set of measure zero.

Proof. Without loss of generality we may suppose $0 < p < 1$.

Let ω be an arbitrary number represented by a power series $\sum_{n=0}^{\infty} a_n p^n$ with integral coefficients a_n, so that

$$|a_n| < \frac{1}{2p} - \frac{1}{2} \qquad (n \geqq 1).$$

If M denotes the largest integer $< \frac{1}{2p} - \frac{1}{2}$, then it follows $|a_n| \leq M$. On the other hand $M \geq 0$, hence $M = \alpha \frac{1-p}{2p}$, where α only depends on p and where $0 \leq \alpha < 1$ is. Hence

$$\left| \sum_{k=m+1}^{\infty} a_k p^k \right| \leqq \alpha \frac{1-p}{2p} \sum_{k=m+1}^{\infty} p^k = \frac{\alpha}{2} p^m \quad (m \geqq 1).$$

Using the notation of § 1 we have by (1) and (2)

$$|\omega_m p^m| = \left| \sum_{k=m+1}^{\infty} a_k p^k \right| \leqq \frac{\alpha}{2} p^m,$$

and it follows

$$|\omega_m| \leqq \frac{\alpha}{2} \quad (m \geqq 1).$$

Let S_m denote the set cf numbers of the form $\sum_{k=0}^{m} a_k p^k$, where the a_k are integers with $|a_k| < \frac{1}{2p} - \frac{1}{2}$ $(k \geq 1)$. We prove by induction that for two different numbers A_m and A'_m of S_m the difference is at least p^m. This is obviously true for $m = 0$. Now let $m \geq 1$ and suppose we have shown the property for $m - 1$ instead of m. Let

$$A_m = \sum_{k=0}^{m} a_k p^k \text{ and } A'_m = \sum_{k=0}^{m} a'_k p^k.$$

Put

$$A_{m-1} = \sum_{k=0}^{m-1} a_k p^k \text{ and } A'_{m-1} = \sum_{k=0}^{m-1} a_k p^k,$$

then

$$A_m = a_m p^m + A_{m-1} \text{ and } A'_m = a'_m p^m + A'_{m-1}.$$

If $A_{m-1} = A'_{m-1}$, it follows

$$|A_m - A'_m| = |a_m - a'_m| \, p^m \geqq p^m.$$

If $A_{m-1} \neq A'_{m-1}$, then

$$|A_{m-1} - A'_{m-1}| \geqq p^{m-1},$$

hence

$$\begin{aligned}|A_m - A'_m| &= |A_{m-1} - A'_{m-1} + (a_m - a'_m)\, p^m| \\ &\geqq |A_{m-1} - A'_{m-1}| - (|a_m| + |a'_m|)\, p^m \\ &\geqq p^{m-1} - 2\left(\frac{1}{2p} - \frac{1}{2}\right) p^m = p^m.\end{aligned}$$

For every number ω and $m \geqq 1$ we can write

$$\omega = A_m + \omega_m\, p^m,$$

where A_m belongs to S_m and $|\omega_m| \leqq \frac{\alpha}{2}$, hence there exists a closed interval

$$\left(A_m - \frac{\alpha}{2}\, p^m,\ A_m + \frac{\alpha}{2}\, p^m\right),$$

which contains ω. Two different intervals

$$\left(A_m - \frac{\alpha}{2}\, p^m,\ A_m + \frac{\alpha}{2}\, p^m\right) \text{ and } \left(A'_m - \frac{\alpha}{2}\, p^m,\ A'_m + \frac{\alpha}{2}\, p^m\right)$$

have for distance at least $p^m - 2 \cdot \frac{\alpha}{2}\, p^m = (1-\alpha)p^m$. Hence all open intervals

$$\left(A_m - p^m + \frac{\alpha}{2}\, p^m,\ A_m - \frac{\alpha}{2}\, p^m\right)$$

and all open intervals

$$\left(A_m + \frac{\alpha}{2}\, p^m,\ A_m + p^m - \frac{\alpha}{2}\, p^m\right)$$

do not contain a number ω.

Let (c, d) denote an interval of length $2p^m$ $(m \geqq 1)$. I shall prove that (c, d) contains a sub-interval of length $(1-\alpha)p^m$, which does not contain a number ω. Without loss of generality we may suppose that (c, d) contains at least one of these numbers, say $\omega^{(0)}$. Then either the interval $(\omega^{(0)}, \omega^{(0)} + p^m)$ or $(\omega^{(0)} - p^m, \omega^{(0)})$ belongs to (c, d). Let

$$\omega^{(0)} = A_m^{(0)} + \omega_m^{(0)}\, p^m \ \textit{with}\ |\omega_m^{(0)}| \leqq \frac{\alpha}{2},$$

$A_m^{(0)}$ belonging to S_m, then

$$A_m^{(0)} + \frac{\alpha}{2}\, p^m \geqq \omega^{(0)} \ \textit{and}\ A_m^{(0)} - \frac{\alpha}{2}\, p_m \leqq \omega^{(0)}.$$

It follows, that the interval i:

$$A_m^{(0)} + \frac{\alpha}{2}\, p^m < x < A_m^{(0)} + p^m - \frac{\alpha}{2}\, p^m$$

of length $(1-\alpha)p^m$ is contained in $(\omega^{(0)}, \omega^{(0)}+p^m)$, and i':

$$A_m^{(0)} - p^m + \frac{\alpha}{2}p^m < x < A_m^{(0)} - \frac{\alpha}{2}p^m$$

belongs to $(\omega^{(0)}-p^m, \omega^{(0)})$. Both intervals i and i' do not contain any number ω and at least one of them is a subinterval of (c, d).

If (e, f) is an interval of length $l < 2p$, then there exists a positive integer $m \geqq 2$ such that $2p^{m-1} > l \geqq 2p^m$. Because $l \geqq 2p^m$ and

$$(1-\alpha)p^m = \frac{1-\alpha}{2}p \cdot 2p^{m-1} > \frac{1-\alpha}{2}pl$$

we have the following result: each interval of length $l < 2p$ contains a subinterval of length $\frac{1-\alpha}{2}pl$, which does not contain a number ω.

Now it easily can be deduced that the set of numbers ω in an arbitrary interval i of length $\lambda < 2p$ has measure zero. Put $\tau = \frac{1-\alpha}{2}p$. Applying our foregoing result we find that all numbers ω in the interval i belong to one or two intervals with total length $(1-\tau)\lambda$. Repeating the process we obtain that all numbers ω in i are also contained in a finite set of intervals with total length $(1-\tau)\lambda - \tau(1-\tau)\lambda = (1-\tau)^2\lambda$. By induction we conclude that for every positive integer n there exists a finite set of intervals with total length $(1-\tau)^n\lambda$, which contains all numbers ω in i. The assertion follows for $n \to \infty$.

§ 3. *If r is a positive integer, if $p_1, p_2, \ldots, p_r$ are different from zero with $-1 < p_1 < p_2 < \ldots < p_r < +1$ and if $\eta_1, \eta_2 \ldots, \eta_r$ are arbitrary real numbers, then there exists a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with bounded integral coefficients a_n, such that $f(p_\varrho) = \eta_\varrho$ $(\varrho = 1, 2, \ldots, r)$.*

Proof. I introduce the following matrices with r rows and r columns

$$\Pi = \begin{pmatrix} p_1 & p_2 \cdots p_r \\ p_1^2 & p_2^2 \cdots p_r^2 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_1^r & p_2^r \cdots p_r^r \end{pmatrix}, \quad P = \begin{pmatrix} p_1^r & 0 \ldots 0 \\ 0 & p_2^r \ldots 0 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 & 0 \ldots p_r^r \end{pmatrix}.$$

These matrices possess an inverse, since $\det P$ and $\det \Pi$ both are different from zero on account of the conditions made for $p_1, p_2, \ldots, p_r$. Further, if α denotes an arbitrary r-dimensional vector with the components $\alpha_1, \alpha_2, \ldots, \alpha_r$, then

$$\alpha \, \Pi \, P^{m-1} \qquad (m = 1, 2, \ldots) \quad \ldots \ldots \quad (3)$$

again is a r-dimensional vector with the components

$$\alpha_1 p_\varrho^{(m-1)r+1} + \alpha_2 p_\varrho^{(m-1)r+2} + \ldots + \alpha_r p_\varrho^{mr} \qquad (\varrho = 1, 2, \ldots, r) \, . \quad (4)$$

Moreover I introduce the notation $|\alpha|$, denoting the maximum of the numbers $|\alpha_1|, |\alpha_2|, \ldots, |\alpha_r|$.

The coefficients of the series $\sum_{n=0}^{\infty} a_n x^n$ are determined in the following manner. I put $a_0 = 0$. In order to define the vector $a^{(1)}$, consisting of the integers $a_1, a_2, \ldots, a_r$, I consider the vector $\omega^{(0)}$ with the components $\eta_1, \eta_2, \ldots, \eta_r$; then $\omega^{(0)}\Pi^{-1}$ represents a r-dimensional vector, which can be written as

$$\omega^{(0)}\, \Pi^{-1} = a^{(1)} + \omega^{(1)} \textit{ with } |\omega^{(1)}| \leqq \tfrac{1}{2}.$$

We have

$$\omega^{(0)} = a^{(1)}\, \Pi + \omega^{(1)}\, \Pi.$$

Now I define the vectors $a^{(1)}, a^{(2)}, \ldots$ by induction: suppose that the r-dimensional vectors $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$; $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(m)}$ are determined such that $m \geqq 1$, $a^{(1)}, a^{(2)}, \ldots, a^{(m)}$ have integral components, $|\omega^{(k)}| \leqq \frac{1}{2}$ for $k = 1, 2, \ldots, m$ and

$$\omega^{(0)} = a^{(1)}\, \Pi + a^{(2)}\, \Pi P + \ldots + a^{(k)}\, \Pi P^{k-1} + \omega^{(k)}\, \Pi P^{k-1} \qquad (k = 1, 2, \ldots, m). \qquad (5)$$

Then I can write the vector $\omega^{(m)} \Pi P^{-1} \Pi^{-1}$ in the form $a^{(m+1)} + \omega^{(m+1)}$, where $a^{(m+1)}$ has integral components and $|\omega^{(m+1)}| \leqq \frac{1}{2}$. It follows

$$\omega^{(m)}\, \Pi P^{m-1} = \omega^{(m)}\, \Pi P^{-1}\, \Pi^{-1} \cdot \Pi P^m = \{a^{(m+1)} + \omega^{(m+1)}\}\, \Pi P^m,$$

hence

$$\omega^{(0)} = a^{(1)}\, \Pi + a^{(2)}\, \Pi P + \ldots + a^{(m+1)}\, \Pi P^m + \omega^{(m+1)}\, \Pi P^m.$$

Successively vectors $a^{(m)}$ *with integers as components and vectors* $\omega^{(m)}$ *with* $|\omega^{(m)}| \leqq \frac{1}{2}$ *can be determined, so that the relation* (5) *is true for* $k = 1, 2, \ldots$.

The vector $\omega^{(m)}\, \Pi P^{m-1}$ has the form (3), hence it follows from (4) that all its components, say $\eta_\varrho^{(m)}$, tend to zero for $m \to \infty$ because $|p_\varrho| < 1$ and $|\omega^{(m)}| \leqq \frac{1}{2}$.

On the other hand each of the vectors $a^{(1)}\Pi$, $a^{(2)}\Pi P, \ldots$ also has the form (3), hence from (4) the components of

$$a^{(1)}\, \Pi + a^{(2)}\, \Pi P + \ldots + a^{(m)}\, \Pi P^{m-1}$$

can be written

$$a_1\, p_\varrho + a_2\, p_\varrho^2 + \ldots + a_{mr}\, p_\varrho^{mr} \qquad (\varrho = 1, 2, \ldots, r),$$

where $a_1, a_2, \ldots, a_{mr}$ are integers $(m \geqq 1)$.

It follows from (5) with $k = m$

$$\eta_\varrho = a_0 + a_1\, p_\varrho + a_2\, p_\varrho^2 + \ldots + a_{mr}\, p_\varrho^{mr} + \eta_\varrho^{(m)} \qquad (\varrho = 1, 2, \ldots, r).$$

Hence

$$\eta_\varrho = a_0 + a_1\, p_\varrho + a_2\, p_\varrho^2 + \ldots + a_{mr}\, p_\varrho^{mr} + \ldots.$$

The matrix $\Pi P^{-1} \Pi^{-1}$ only depends on $p_1, p_2, \ldots, p_r$; since $|\omega^{(m)}| \leqq \frac{1}{2}$ $(m \geqq 1)$, it follows that the vectors $a^{(m+1)} + \omega^{(m+1)} = \omega^{(m)}\, \Pi P^{-1}\, \Pi^{-1}$ are bounded; the same is true for the vectors $a^{(m)}$ for $m \geqq 2$; hence all vectors $a^{(m)}$ are bounded.

Aerodynamics. — *On the Appearance of Vortex Movements in the Sun.*

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(Communicated by Prof. J. M. BURGERS.)

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1. *Introduction.* — Certain movements of sunspots might represent vortices connected with the variation of the solar velocity of rotation with heliographic latitude (*1*). The present investigation has been instigated by the appearance of such movements. Its purpose is to study forms of motion in the sun on the basis of the equations of motion of a fluid. The investigation will be confined to the outer parts of the sun, where the results can be checked by observation. Obviously we must take into account the compressibility of the matter. As only slow movements are considered, the equation of adiabacy is not applicable, but the temperature distribution, similarly as the gravitational potential, will be assumed to be unaffected by the motion. As usual in investigations of this kind, the effect of viscosity will first be neglected. It cannot be known in advance how accurately the undisturbed rotation and the distribution of the density and temperature within the sun should be known. Therefore we shall go immediately to the problem of the disturbed motion and then assume a polytropic distribution of the density. In any case the distribution of the undisturbed velocity at the surface is known from direct observations of the velocity of rotation of the sun.

2. *The Equations of Motion.* — We use LAMB's (*2*) notation for spherical polar coordinates ($\theta =$ North polar distance, $\omega =$ longitude, $r =$ distance from the centre) and velocities (u radial, v N—S, w E—W component), and write P, ϱ for pressure and density, respectively. The equations of motion which are sufficiently well known will not be repeated here (*2*). The pressure is determined by the equation of state

$$P = \Re \varrho T/\mu,$$

where T is the temperature, $\Re$ the absolute gas constant and μ the mean molecular weight of the matter. Insert into the equations:

$$u = u'; \quad v = v'; \quad w = W + w'; \quad P = P_0 + \tilde{\omega}; \quad \varrho = \varrho_0 + \sigma,$$

where W, P_0, ϱ_0 are the values for the undisturbed rotation, and assume that terms of the second order in the quantities $u' \dots \sigma$ and their derivatives may be neglected. Dropping the primes and subscripts two sets of equations are then obtained, of which we give the set referring to the disturbed motion [1]):

[1]) Equations (2. 1)—(2. 3) are the equations of motion while (2. 4) is the equation of continuity.

$$\frac{\partial u}{\partial t}+\frac{W}{r\sin\theta}\frac{\partial u}{\partial\omega}-2\frac{W}{r}w=\frac{\sigma}{\varrho^2}\frac{\partial P}{\partial r}-\frac{1}{\varrho}\frac{\partial\tilde{\omega}}{\partial r}\quad\ldots\quad(2.1)$$

$$\frac{\partial v}{\partial t}+\frac{W}{r\sin\theta}\frac{\partial v}{\partial\omega}-2\frac{W}{r}w\cot\theta=\frac{\sigma}{\varrho^2}\frac{1}{r}\frac{\partial P}{\partial\theta}-\frac{1}{\varrho}\frac{1}{r}\frac{\partial\tilde{\omega}}{\partial\theta}\quad\ldots\quad(2.2)$$

$$\frac{\partial w}{\partial t}+\frac{W}{r\sin\theta}\frac{\partial w}{\partial\omega}+\left(\frac{\partial W}{\partial r}+\frac{W}{r}\right)u+\left(\frac{\partial W}{\partial\theta}+W\cot\theta\right)\frac{v}{r}=-\frac{1}{\varrho r\sin\theta}\frac{\partial\tilde{\omega}}{\partial\omega}\quad(2.3)$$

$$\frac{\partial\sigma}{\partial t}+\frac{W}{r\sin\theta}\frac{\partial\sigma}{\partial\omega}+\frac{2}{r}\varrho u+\frac{\cot\theta}{r}\varrho v+\frac{1}{r\sin\theta}\frac{\partial(\varrho w)}{\partial\omega}+$$
$$+\frac{\partial(\varrho u)}{\partial r}+\frac{1}{r}\frac{\partial(\varrho v)}{\partial\theta}=0.\quad(2.4)$$

Further

$$\tilde{\omega}=\Re T\sigma/\mu=P\sigma/\varrho.\quad\ldots\quad(2.5)$$

We shall investigate whether equations (2, 1)—(2, 5) admit a particular solution of the form

$$\left.\begin{aligned}\varrho r\sin\theta\cdot u&=A\cos(n\omega-\nu t)\\ \varrho r\sin\theta\cdot v&=B\cos(n\omega-\nu t)\\ \varrho r\sin\theta\cdot w&=C\sin(n\omega-\nu t)\\ \sigma&=\varrho D\sin(n\omega-\nu t)\\ \tilde{\omega}&=PD\sin(n\omega-\nu t)\end{aligned}\right\}\quad\ldots\quad(2.6)$$

where A, B, C and D are supposed to be functions of r and θ only. With the notation

$$Wr\sin\theta=H\quad\ldots\quad(2.7)$$

and

$$-\nu+\frac{W}{r\sin\theta}n=-\nu+\frac{H}{r^2\sin^2\theta}n=\lambda\quad\ldots\quad(2.8)$$

the following system of equations is obtained:

$$\lambda A+\frac{2H}{r^2\sin\theta}C=rP\sin\theta\cdot D_r\quad\ldots\quad(2.9)$$

$$\lambda B+\frac{2H\cos\theta}{r^2\sin^2\theta}C=P\sin\theta\cdot D_\theta\quad\ldots\quad(2.10)$$

$$\lambda C+\frac{H_r}{r\sin\theta}A+\frac{H_\theta}{r^2\sin\theta}B=-P\cdot nD\quad\ldots\quad(2.11)$$

$$\lambda D+\frac{1}{\varrho r^2\sin\theta}A+\frac{n}{\varrho r^2\sin^2\theta}C+\frac{1}{\varrho r\sin\theta}A_r+\frac{1}{\varrho r^2\sin\theta}B_\theta=0.\quad(2.12)$$

Here the subscripts r and θ denote derivatives with respect to r and θ. It is possible to derive from these equations a single differential equation of the second order for D.

Let us denote the determinant of equations (2, 9)—(2, 11) by $\lambda \triangle$. Then:

$$\triangle = \lambda^2 - \frac{2\,HH_r}{r^3 \sin^2\theta} - \frac{2\,HH_\theta \cos\theta}{r^4 \sin^3\theta} \quad \ldots \ldots \quad (2.\,13)$$

If λ or $\triangle$ vanishes, equations (2, 9)—(2, 11) can have a solution only if $n\,D$, $\sin\theta \cdot D_r$ and $\sin\theta \cdot D_\theta$ vanish likewise. When this is not the case, we have:

$$A = \frac{rP\sin\theta}{\lambda\triangle}\left(\lambda^2 - \frac{2\,HH_\theta\cos\theta}{r^4\sin^3\theta}\right) D_r + \frac{2\,HH_\theta\,P}{\lambda\triangle r^4 \sin\theta} D_\theta + \frac{2\,HP}{\triangle r^2 \sin\theta} nD \quad (2.\,14)$$

$$B = \frac{2\,HH_r P\cos\theta}{\lambda\triangle r^2 \sin^2\theta} D_r + \frac{P\sin\theta}{\lambda\triangle}\left(\lambda^2 - \frac{2\,HH_r}{r^3\sin^2\theta}\right) D_\theta + \frac{2\,HP\cos\theta}{\triangle r^2\sin^2\theta} nD \quad (2.\,15)$$

$$C = -\frac{H_r P}{\triangle} D_r - \frac{H_\theta P}{\triangle r^2} D_\theta - \frac{\lambda P}{\triangle} nD. \quad \ldots \quad (2.\,16)$$

When these expressions are inserted in equation (2, 12) the following differential equation for D is obtained:

$$\left.\begin{aligned}
&\left(\lambda^2 - \frac{2\,HH_\theta\cos\theta}{r^4\sin^3\theta}\right) D_{rr} + \frac{2\,H}{r^4\sin^2\theta}\left(\frac{H_\theta}{r} + H_r\cot\theta\right) D_{r\theta} + \frac{1}{r^2}\left(\lambda^2 - \frac{2\,HH_r}{r^3\sin^2\theta}\right) D_{\theta\theta} + \\
&+\left\{\frac{\lambda\triangle}{Pr\sin\theta}\frac{\partial}{\partial r}\left[\frac{rP\sin\theta}{\lambda\triangle}\left(\lambda^2 - \frac{2\,HH_\theta\cos\theta}{r^4\sin^3\theta}\right)\right] + \frac{\lambda\triangle}{Pr^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{2\,HH_r P\cos\theta}{r^2\sin^2\theta\lambda\triangle}\right)\right. \\
&\qquad\left. + \frac{1}{r}\left(\lambda^2 - \frac{2\,HH_\theta\cos\theta}{r^4\sin^3\theta}\right) + \frac{\lambda n}{r^2\sin^2\theta}\left(\frac{2\,H}{r} - H_r\right)\right\} D_r + \\
&+\left\{\frac{\lambda\triangle}{Pr\sin\theta}\frac{\partial}{\partial r}\left(\frac{2\,HH_\theta P}{r^4\sin\theta\lambda\triangle}\right) + \frac{\lambda\triangle}{Pr^2\sin\theta}\frac{\partial}{\partial\theta}\left[\frac{P\sin\theta}{\lambda\triangle}\left(\lambda^2 - \frac{2\,HH_r}{r^3\sin^2\theta}\right)\right]\right. \\
&\qquad\left. + \frac{2\,HH_\theta}{r^6\sin^2\theta} + \frac{n\lambda}{r^4\sin^2\theta}(2\,H\cot\theta - H_\theta)\right\} D_\theta + \\
&+\left\{\frac{\lambda\triangle}{Pr\sin\theta}\frac{\partial}{\partial r}\left(\frac{2\,nHP}{\triangle r^2\sin\theta}\right) + \frac{\lambda\triangle}{Pr^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{2\,nHP\cos\theta}{\triangle r^2\sin^2\theta}\right)\right. \\
&\qquad\left. + \frac{2\,Hn\lambda}{r^4\sin^2\theta} - \frac{n^2\lambda^2}{r^2\sin^2\theta} + \frac{\varrho\lambda^2\triangle}{P}\right\} D = 0.
\end{aligned}\right\} \quad (2.\,17)$$

3. *The Assumed Law of Solar Rotation. Equation for D Expressed in Dimensionless Quantities.* — It seems rather appropriate to assume, when the vicinity of the axis is excluded, that points at the same distance from the axis have the same undisturbed velocity (*3*). With $a = 0.237$, $b = 0.24$ and R the solar radius, we then have:

$$W = a\,[1 + b\,(r/R)^2 \sin^2\theta] \cdot 10^{-5}\, r\sin\theta \quad \ldots \ldots \quad (3.\,1)$$

$$H = a\,[1 + b\,(r/R)^2 \sin^2\theta] \cdot 10^{-5}\, r^2\sin^2\theta \quad \ldots \ldots \quad (3.\,2)$$

$$\lambda = -\nu + na\,[1 + b\,(r/R)^2 \sin^2\theta] \cdot 10^{-5} \quad \ldots \ldots \quad (3.\,3)$$

$$\triangle = \lambda^2 - \frac{2\,HH_\theta}{r^4 \sin^3\theta \cos\theta} = \lambda^2 - \frac{2\,HH_r}{r^3 \sin^4\theta} \quad . \quad . \quad . \quad . \qquad (3.4)$$

$$r H_r = H_\theta \tan\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3.5)$$

To simplify equation (2, 17) we now introduce the following expressions:

$$10^{-5}\,a = \Omega; \qquad r/R = x,$$
$$1 + bx^2 \sin^2\theta = \psi; \qquad 1 + 2\,bx^2 \sin^2\theta = \psi^*,$$
$$4\,\psi\psi^*/n^2 = \xi; \qquad \psi - \nu/n\Omega = \varphi; \qquad \lambda = n\Omega \cdot \varphi.$$

Then

$$H = \Omega R^2 \cdot \psi x^2 \sin^2\theta$$
$$H_x = \Omega R^2 \cdot 2\,\psi^* x \sin^2\theta$$
$$H_\theta = \Omega R^2 \cdot 2\,\psi^* x^2 \sin\theta\cos\theta$$
$$\triangle = n^2\Omega^2 (\varphi^2 - \xi)$$
$$\lambda^2 - \frac{2\,H H_\theta \cos\theta}{r^4 \sin^3\theta} = n^2\Omega^2 (\varphi^2 - \xi\cos^2\theta)$$
$$\lambda^2 - \frac{2\,HH_r}{r^3 \sin^2\theta} = n^2\Omega^2 (\varphi^2 - \xi\sin^2\theta).$$

Further we write

$$\frac{x}{P}\frac{\partial P}{\partial x} = \Pi; \qquad \frac{1}{P}\frac{\partial P}{\partial\theta} = \Pi^*.$$

After division by $n^2\Omega^2/R^2$ equation (2, 17) then obtains the form:

$$\left.\begin{aligned}
&(\varphi^2 - \xi\cos^2\theta)\,D_{xx} + \frac{2\,\xi\sin\theta\cos\theta}{x}\,D_{x\theta} + \frac{\varphi^2 - \xi\sin^2\theta}{x^2}\,D_{\theta\theta} + \\
&+ \Big\{(\varphi^2 - \xi\cos^2\theta)\left[\frac{\Pi + 2}{x} + \varphi(\varphi^2 - \xi)\frac{\partial}{\partial x}\left(\frac{1}{\varphi(\varphi^2 - \xi)}\right) + \frac{\partial}{\partial x}(\varphi^2 - \xi\cos^2\theta)\right. \\
&\qquad + \frac{\xi\sin\theta\cos\theta}{x}\,\Pi^* + \frac{\varphi(\varphi^2 - \xi)}{x\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\xi\sin^2\theta\cos\theta}{\varphi(\varphi^2 - \xi)}\right) - 2\,b\varphi x\sin^2\theta\Big\}\,D_x + \\
&+ \Big\{\frac{\xi\sin\theta\cos\theta}{x^2}\,\Pi + \frac{\varphi(\varphi^2 - \xi)\sin\theta\cos\theta}{x}\frac{\partial}{\partial x}\left(\frac{\xi}{\varphi(\varphi^2 - \xi)}\right) + \frac{\varphi^2 - \xi\sin^2\theta}{x^2}\,\Pi^* \\
&\qquad + \frac{\varphi(\varphi^2 - \xi)}{x^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta(\varphi^2 - \xi\sin^2\theta)}{\varphi(\varphi^2 - \xi)}\right) + \frac{\xi\sin\theta\cos\theta}{x^2} - 2\,b\varphi\sin\theta\cos\theta\Big\}\,D_\theta + \\
&+ \Big\{\frac{2\,\psi\varphi}{x^2}(\Pi + 1) + \frac{2\,\psi\varphi\cos\theta}{x^2\sin\theta}\,\Pi^* + \frac{2\,\varphi(\varphi^2 - \xi)}{x\sin\theta}\left[\frac{\partial}{\partial x}\left(\frac{\psi\sin\theta}{\varphi^2 - \xi}\right) + \frac{1}{x}\frac{\partial}{\partial\theta}\left(\frac{\psi\cos\theta}{\varphi^2 - \xi}\right)\right] \\
&\qquad - \frac{n^2\varphi^2}{x^2\sin^2\theta} + \frac{\mu R^2\Omega^2}{\Re T}\,n^2\varphi^2(\varphi^2 - \xi)\Big\}\,D = 0.
\end{aligned}\right\} \quad (3.6)$$

4. *Stationary Solutions with Large n. The Standard Model.* — In order to obtain an idea of the solutions of this complicated equation we shall now limit our attention to stationary solutions ($\nu = 0$) which further are characterized by large values of n. It should be observed that these solutions are stationary with respect to coordinate axes fixed in space. In a coordinate system rotating with the sun the motion is periodic with period T_0/n, where T_0 is the period of the solar rotation. It is upon this quantity (T_0/n) that the possibility of adaptation of the temperature will depend: if n is too large, the temperature cannot adjust itself and conditions will more approach to those found in adiabatic motion.

It should further be kept in mind that when the solution gives a finite value of the radial component of the velocity, periodic in ω, at the surface of the sun, this means that the matter of the sun's surface has a periodic radial motion, which for an observer rotating with the sun will have the character of a wave system. (An observer fixed in space will see a pattern of velocities which does not change with time.)

To fix ideas we assume n to have a value between 30 and 100. Then $\varphi = \psi$ is of the order unity, while ξ is of order n^{-2} and will be neglected when it appears in conjunction with quantities of order unity. After division by $\varphi^2 = \psi^2$ we then find:

$$\left.\begin{aligned}
&D_{xx} + \frac{2\xi\sin\theta\cos\theta}{\psi^2 x} D_{x\theta} + \frac{1}{x^2} D_{\theta\theta} + \\
&+ \left\{ \frac{\Pi+2}{x} - \frac{1}{\psi}\frac{\partial\psi}{\partial x} + \frac{\xi\sin\theta\cos\theta}{\psi^2 x}\Pi^* + \frac{\psi}{x\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\xi\sin^2\theta\cos\theta}{\psi^3}\right) \right. \\
&\qquad\qquad \left. - 2b\,\frac{x\sin^2\theta}{\psi} \right\} D_x + \\
&\quad + \left\{ \frac{\xi\sin\theta\cos\theta}{\psi^2 x^2}\Pi + \frac{\psi\sin\theta\cos\theta}{x}\frac{\partial}{\partial x}\left(\frac{\xi}{\psi^3}\right) + \frac{1}{x^2}\Pi^* \right. \\
&\quad \left. + \frac{\psi}{x^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\psi}\right) + \frac{\xi\sin\theta\cos\theta}{\psi^2 x^2} - 2b\,\frac{\sin\theta\cos\theta}{\psi} \right\} D_\theta + \\
&+ \left\{ \frac{2}{x^2}\left(\Pi+1\right) + \frac{2\cos\theta}{x^2\sin\theta}\Pi^* + \frac{2\psi}{x\sin\theta}\left[\frac{\partial}{\partial x}\left(\frac{\sin\theta}{\psi}\right) + \frac{1}{x}\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\psi}\right)\right] \right. \\
&\qquad\qquad \left. - \frac{n^2}{x^2\sin^2\theta} + \frac{\mu R^2\Omega^2}{\Re T} n^2\psi^2 \right\} D = 0.
\end{aligned}\right\} \quad (4.\,1)$$

In this equation we can immediately neglect the term with $D_{x\theta}$. Its unimportance can be proved for any expression of the form

$$D_{xx} + \alpha D_{x\theta} + \beta D_{\theta\theta}$$

where $\alpha^2/4 \ll \beta$ (as is the case in our equation), by making the substitution $\xi = x$, $\eta = \theta - \frac{1}{2}\alpha x$. Then the second term vanishes and the third term obtains a vanishingly small correction.

The terms of the second order thus reduce to:

$$D_{xx} + \frac{1}{x^2} D_{\theta\theta},$$

which are of the same order of magnitude.

When next we look at the factor of D we see that one of the terms of this factor, $n^2/x^2 \sin^2\theta$, will be large, in consequence of the assumption concerning n. Also Π appears to be a large quantity, as will be shown subsequently. On the other hand in the outer parts of the sun the condition $\Pi^* \ll \Pi$ is undoubtedly satisfied in such a way what even

$$\Pi^*/\sin\theta \ll \Pi.$$

Including the most important terms only, the factor of D thus reduces to:

$$\frac{2}{x^2}\Pi - \frac{n^2}{x^2 \sin^2\theta} + \frac{\mu R^2 \Omega^2}{\Re T} n^2 \psi^2.$$

In order to make further estimates of the various terms, we calculate the quantity Π assuming that in the outer part of the sun the pressure, density and temperature depend on x similarly as in the standard model (polytropic model of index 3). We put $T = T_c \cdot y$, where T_c is the temperature at the centre of the sun (about $2 \cdot 10^7$, corresponding to $\mu \sim 1$). As the surface temperature is approximately 5000, the value of y at the surface will be $5 \cdot 10^3/2 \cdot 10^7 = 2.5 \cdot 10^{-4}$. In the outer part of the standard model we further have with sufficient accuracy

$$y = 0.2926\ (1-x), \quad \ldots \ldots \ldots \quad (4.\ 2)$$

so that the surface is determined by $(1-x)_0 = 8.6 \cdot 10^{-4}$. Further in the standard model $P =$ const. y^4. Hence,

$$\Pi = \frac{x}{P}\frac{\partial P}{\partial x} = 4\frac{x}{y}\frac{dy}{dx},$$

or, at the surface, $-\Pi = 4700$. At $x = 0.8$, $-\Pi$ is found to have the value 19.7, from which the value of $-\Pi$ increases outwards.

Coming now to the factors of D_x and D_θ it is immediately seen that in the factor of D_x we need to retain only the term Π/x, which far exceeds all other terms. In the factor of D_θ the only terms which may be of some importance are

$$\frac{\xi \sin\theta \cos\theta}{\psi^2 x^2}\Pi \text{ and } \frac{\cot\theta}{x^2},$$

the latter because it approaches infinity at the poles.

After these simplifications the following differential equation is obtained:

$$\left.\begin{aligned} D_{xx} + \frac{1}{x^2} D_{\theta\theta} + \frac{\Pi}{x} D_x + \left\{\frac{\xi \sin\theta\cos\theta}{\psi^2 x^2}\Pi + \frac{\cot\theta}{x^2}\right\} D_\theta + \\ + \left\{\frac{2}{x^2}\Pi - \frac{n^2}{x^2\sin^2\theta} + \frac{\mu R^2\Omega^2}{\Re T} n^2\psi^2\right\} D = 0. \end{aligned}\right\} \quad (4.\ 3)$$

The first term in the factor of D_θ, however, may still be neglected if we compare it with the factor of $D_{\theta\theta}$ and the first term in the factor of D. Indeed, if in an expression $\alpha D_{\theta\theta} + \beta D_\theta + \gamma D$, D is replaced by e^η, it can be proved that βD_θ may be neglected in comparison either with $\alpha D_{\theta\theta}$ or with γD, provided $\beta^2 \ll \alpha\gamma$. In the case of equation (4, 3) this condition amounts to

$$\xi^2 \sin^2\theta \cos^2\theta \cdot |\Pi| \ll 2\psi^4.$$

As $\psi \sim 1$, $\xi^2 = (4\psi\psi^*/n^2)^2 \sim 16\,n^{-4}$, $\sin^2\theta\cos^2\theta \leq 1/4$, and $|\Pi| \leq 4700$, the condition is satisfied when $n > 30$.

The same argument might be applied also to the term $\cot\theta/x^2$. However, we have retained this term, since it does not introduce any difficulty in the solution of the equation.

Let us now compare with each other the three remaining terms in the factor of D. Choosing $\theta = \pi/2$, so that the second term will obtain its smallest possible value while the third term obtains its largest possible value (because of ψ), the ratio of the three terms will be at the surface:

$$9400 : n^2 : 0.10\,n^2.$$

Having regard to the accuracy of observations, it can be said that the third term is small compared with the second one. Moreover, when we keep in mind that the third decreases rapidly when we go inwards, it can certainly be neglected. (The third term varies proportionally to the expression $1/y = 1/[0.2926(1-x)]$, where at the surface $1 - x = 8.6 \cdot 10^{-4}$). The remaining terms will then be the first and the second, which are of the same order of magnitude and have the same sign. The first of them decreases rapidly inwards.

The final form of equation (3, 6) will now be

$$D_{xx} + \frac{1}{x^2} D_{\theta\theta} + \frac{\Pi}{x} D_x + \frac{\cot\theta}{x^2} D_\theta + \left\{ \frac{2}{x^2}\Pi - \frac{n^2}{x^2 \sin^2\theta} \right\} D = 0. \quad (4.4)$$

In this equation the differential rotation as well as the velocity of the rotation itself seem to have disappeared. However, if we assume the velocity of rotation to vanish, the first term in the brackets will vanish. On the other hand, the non-appearance of the differential rotation does not necessarily mean that the movements which follow from equation (4, 4) could not represent movements excited by the differential rotation, as amongst the various possible forms of excited movements there may be components representing stationary movements corresponding to solutions of (4, 4) with n between 30 and 100.

Along with equation (4, 4), D has to satisfy certain *boundary conditions*. Such conditions will be the following:

1. The quantity D remains finite for any value of θ.
2. The quantity D tends to zero inwards but remains finite even at the surface.

The first of these conditions is evident. To the second we can add that this investigation is confined to such movements only which occur specifically near the surface. If now D would not tend to zero when the pressure P increases, the velocity components A, B, C [equations (2, 14)—(2, 16)] would increase very rapidly inwards. The corresponding movements would not be specifically limited to the vicinity of the surface. On the contrary, they would only represent traces of much stronger movements within the sun. If, however, D is chosen to vanish within the sun, it will be found later that it approaches infinity at $x = 1$. At the observed surface, however, it remains finite. On the other hand D will increase in such a way in the vicinity of the surface that the mass-velocities in any case remain finite even if we go as far as $x = 1$.

Because in the outer layers P may be considered as a function of x only, the variables in equation (4, 4) can be separated by the substitution

$$D = \Phi(x)\,\Psi(\theta) \quad \text{. . . . .} \quad (4.5)$$

The two resulting equations will be:

$$\frac{\Psi''}{\Psi} + \cot\theta\,\frac{\Psi'}{\Psi} - \frac{n^2}{\sin^2\theta} = -m(m+1) \quad \text{. . . .} \quad (4.6)$$

$$x^2\,\frac{\Phi''}{\Phi} + \frac{x^2}{P}\frac{dP}{dx}\,\frac{\Phi'}{\Phi} + \frac{2x}{P}\frac{dP}{dx} = +m(m+1). \quad \text{. . .} \quad (4.7)$$

The quantity $m\,(m+1)$ is a constant of integration.

5. *The Equation for* $\Psi(\theta)$. — The substitution $z = \cos\theta$ brings equation (4, 6) to the form

$$(1-z^2)\,\Psi_{zz} - 2z\,\Psi_z + \left\{ m(m+1) - \frac{n^2}{1-z^2} \right\} \Psi = 0 \quad \text{. .} \quad (5.1)$$

the fundamental solutions of which are FERRERS' (*4*) associated Legendre functions $P_m^n(z)$ and $Q_m^n(z)$. The solutions $Q_m^n(z)$ approach infinity at the equator and may accordingly be excluded. Hence

$$\Psi = \sin^n\theta\,\frac{d^n P_m(\cos\theta)}{(d\cos\theta)^n} = \sin^n\theta \cdot P_m^n(\cos\theta). \quad \text{. . . .} \quad (5.2)$$

Excluding temporarily the S-pole, $P_m(\cos\theta)$ can be expressed by means of the following hypergeometric series [(*4*), p. 311]:

$$P_m(\cos\theta) = F\,[m+1,\ -m,\ 1,\ \tfrac{1}{2}(1-\cos\theta)].$$

Hence, dropping constant factors,

$$P_m^n(\cos\theta) = F\,[m+n+1,\ -m+n,\ n+1,\ \tfrac{1}{2}(1-\cos\theta)].$$

Using this expression for $P_m^n(\cos\theta)$ in equation (5, 2), it can be proved that

$$m = n + \eta \qquad (\eta = 0,\ 1,\ 2,\ 3,\ \ldots)$$

If this condition is not fulfilled, Ψ will approach infinity in the vicinity of the S-pole. After a simple reduction we find:

$$\Psi = \sin^n\theta \cdot F\,[2n+\eta+1,\ -\eta,\ n+1,\ \tfrac{1}{2}(1-\cos\theta)]. \quad (5.3)$$

The functions F contain $\eta + 1$ terms. It can be easily verified that all solutions with even η are symmetrical, while all solutions with odd η are antisymmetrical. The functions Ψ are oscillating functions and η indicates the number of zeros between the N-pole and the S-pole. The amplitude is $\sin^n\theta$; it is vanishingly small at the poles and attains a sharp maximum at the equator.

For numerical calculation it will be more convenient to develop the functions F in powers of $\cos\theta$. For the first few values of η we find:

$$\left.\begin{aligned}
\Psi_0 &= \sin^n\theta \\
\Psi_1 &= \sin^n\theta\cos\theta \\
\Psi_2 &= \sin^n\theta\,[1-(2n+3)\cos^2\theta] \\
\Psi_3 &= \sin^n\theta\cos\theta\,[1-\tfrac{1}{3}(2n+5)\cos^2\theta] \\
\Psi_4 &= \sin^n\theta\,[1-2\,(2n+5)\cos^2\theta+\tfrac{1}{3}(2n+5)(2n+7)\cos^4\theta] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad (5.4)$$

From these expressions the corresponding derivatives are easily found. For instance:

$$\Psi_4' = \left\{ n\frac{\Psi_4}{\sin\theta} + 4\,(2n+5)\sin^{n+1}\theta\,[1-\tfrac{1}{3}(2n+7)\cos^2\theta]\right\}\cos\theta. \quad (5.5)$$

Equation (5, 3) shows that in general Ψ' is greater than Ψ. Their ratio, however, is not of the order n, because various terms in Ψ' counterbalance each other.

6. *The Equation for $\Phi(x)$.* — In order to solve equation (4, 7), P has to be known as a function of x. We shall temporarily assume that the outer part of the sun has a polytropic constitution with a general polytropic index p, and shall afterwards go to the standard model with $p = 3$. Then $P = \text{const.}\ y^{p+1}$ and, approximately, $y = -(1-x)\dfrac{dy}{dx}$. Hence

$$\frac{1}{P}\frac{dP}{dx} = -\frac{p+1}{1-x}.$$

When this is inserted in equation (4, 7), the following equation is obtained:

$$\Phi'' + \frac{p+1}{x-1}\,\Phi' + \left\{\frac{m\,(m+1)}{x} + [2(p+1) - m\,(m+1)]\right\}\frac{\Phi}{x\,(x-1)} = 0. \quad (6.1)$$

This is a hypergeometric differential equation (5). The solution which does not approach infinity at $x = 0$ is given by

$$\left.\begin{aligned}
\Phi = x^{m+1}(1-x)^{-p}\,F\Big(&m+1-\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{4m\,(m+1)+p^2-8\,(p+1)}, \\
&m+1-\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{4m\,(m+1)+p^2-8\,(p+1)},\ 2m+2,\ x\Big)
\end{aligned}\right\} \quad (6.2)$$

As m is large ($m \geqq n$), by putting $p = 3$ it can easily be verified that F is roughly equal to $1 - x$. At $x = 1$, however, it does not vanish but obtains a value approximately equal to $1/m$. The function varies slightly more rapidly near $x = 1$ than near $x = 0$, while the second and higher derivatives at $x = 1$ are nearly zero. In the Taylor series of F according to powers of $(1 - x)$ we may accordingly neglect quadratic and higher terms [2]). Dropping the common denominator we then obtain

$$F = \Gamma(p)\,\Gamma(2m+2) + \frac{m(p-1)-(p+3)}{2m+2}\,\Gamma(p-1)\,\Gamma(2m+3)\,(1-x),$$

or, again dropping a factor and putting $p = 3$,

$$F = (1-x) + \frac{1}{m-3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.3)$$

7. *The Velocity Components.* — We now introduce the expressions of Φ and Ψ and $P = \text{const.}\ (1-x)^{p+1}$ into the equations (2, 14)—(2, 16), neglecting the small terms depending on the differential rotation. It is then convenient to use the following expressions:

$$f = \frac{1}{x}(1-x)^{p+1}\,\Phi; \qquad f^* = (1-x)^{p+1}\,\Phi',$$

or, with $p = 3$, Φ from equation (6, 2) and F from equation (6, 3):

$$f = x^m (1-x)\left[(1-x) + \frac{1}{m-3}\right] \quad . \quad . \quad . \quad . \quad . \quad (7.1)$$

$$f^* = x^m \left\{ [(m+1)(1-x) + 3x]\left[(1-x) + \frac{1}{m-3}\right] - x(1-x) \right\} \quad (7.2)$$

We further write:

$$A^* = A/r \sin\theta; \qquad B^* = B/r \sin\theta; \qquad C^* = C/r \sin\theta. \quad . \quad (7.3)$$

Then we obtain:

$$\left.\begin{aligned}
&(rad.) && A^* = && f^*\Psi + \frac{4\sin\theta\cos\theta}{n^2}\cdot f\Psi' + 2\cdot f\Psi \\
&(N-S) && B^* = && \frac{4\sin\theta\cos\theta}{n^2}\cdot f^*\Psi + f\Psi' + 2\cot\theta\cdot f\Psi \\
&(E-W) && C^* = && -\frac{2\sin\theta}{n}\cdot f^*\Psi - \frac{2\cos\theta}{n}\cdot f\Psi' - \frac{n}{\sin\theta}\cdot f\Psi
\end{aligned}\right\} \quad (7.4)$$

The mass-velocities will then be

$$\varrho u = A^* \cos n\omega; \quad \varrho v = B^* \cos n\omega; \quad \varrho w = C^* \sin n\omega. \quad . \quad (7.5)$$

Further

$$\sigma/\varrho = \Phi\,\Psi \sin n\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.6)$$

---

[2]) The development of Φ in hypergeometric functions around $x = 1$ would contain logarithmic terms when p is an integer.

From (7, 1) and (7, 2) we see that f and f^* vanish very rapidly when we go towards the interior of the sun. f vanishes also at $x = 1$. f^* does not vanish at $x = 1$, but remains finite there. Accordingly the mass-velocities vanish at some distance below the surface. At the surface they approach a finite value. f^* is on the average about $m = n + \eta$ times greater than f.

In order to find out to what temperatures the values of the velocities correspond, we use equation (4, 2), with $T = T_c \cdot y$ and $T_c \sim 2 \cdot 10^7$ degrees. Then we find for the outer part:

$$T \sim 6.10^6 (1-x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.7)$$

8. *Numerical Example.* — Let us now apply the above equations to the case $n = 50$, $\eta = 4$. Before making numerical calculations, we observe that $\Psi' < n\Psi$, except when Ψ is nearly zero, and that generally $f^* > nf$. (At the surface we even have $f^* \sim n^2 f$, as will be seen from the numerical calculations.) Accordingly the second and third terms in A^* may be neglected when compared with the first term.

In order to estimate which are the most important terms in B^* and C^*, we look at the values of $10000 \cdot f$ and $10000 \cdot f^*$ given in Table 1 as

TABLE 1.

$1-x$	T (°K)	$10000 \cdot f$	$10000 \cdot f^*$	$1-x$	T (°K)	$10000 \cdot f$	$10000 \cdot f^*$
0.0000	0	0.000	588	0.005	30000	0.939	574
0.0001	600	0.020	588	0.01	60000	1.721	548
0.0002	1200	0.039	588	0.02	120000	2.661	472
0.0005	3000	0 098	587	0.05	300000	2.182	215
0.001	6000	0.195	586	0.1	600000	0.405	30
0.002	12000	0.388	584	0.2	1200000	0.003	0

functions of $1 - x$ and the temperature, and at the values of Ψ and Ψ' described by Fig. 1. Table 1 shows first that there is very little movement up to about 0.9 of the solar radius. When we keep to values of x corresponding to the observed surface layers, the main term in B^* is the second term. This can be understood from Fig. 1 and Table 1, because Ψ' varies between about -20 and $+20$, while $f^*/f \sim n^2$. (In the surface layers the temperature is about 6000 degrees. Hence $1 - x = 0.001$, and accordingly $f^* \sim 3000 f$.) Similarly we see that C^* is determined by its first and last term. Hence, roughly:

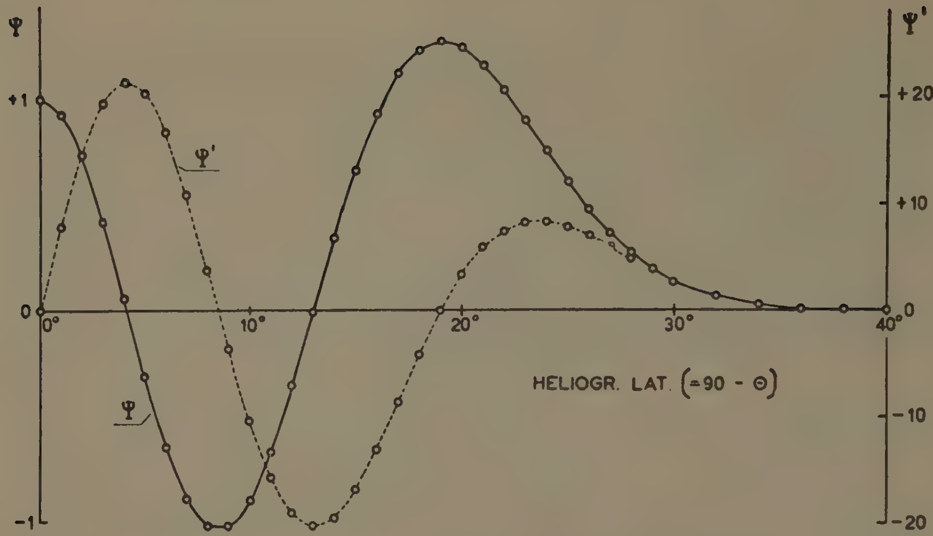
$$A^* \sim f^* \Psi; \quad B^* \sim f \Psi'; \quad C^* \sim -\left(\frac{2 \sin \theta}{n} f^* + \frac{n}{\sin \theta} f\right) \Psi.$$

In the zone where Ψ and Ψ' have their zeros, $\sin \theta \sim 1$. Accordingly, in the surface layers, in this zone:

$$A^* \sim n^2 \Psi; \quad B^* \sim \Psi'; \quad C^* \sim -3n\Psi \quad . \quad . \quad . \quad . \quad (8.1)$$

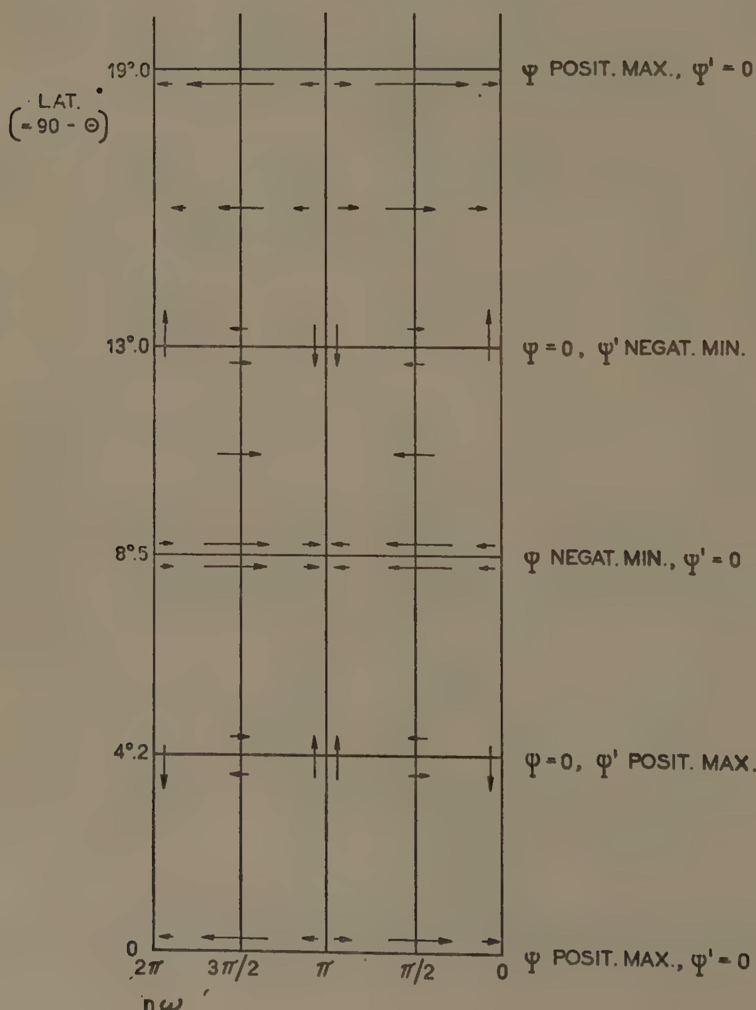
which are valid for $n \sim 50$, $\eta = 4$. Further we obtain:

$$\varrho u \sim n^2 \Psi \cos n\omega; \quad \varrho v \sim \Psi' \cos n\omega; \quad \varrho w \sim -3n\Psi \sin n\omega. \quad (8.2)$$



Ψ
+1
0
-1
Ψ'
Ψ
0°
10°
20°
30°
40°
HELIOGR. LAT. (=90 - Θ)
Ψ'
+20
+10
0
-10
-20

Fig. 1.



LAT. (=90 - Θ)
19°.0
13°.0
8°.5
4°.2
0
2π
3π/2
π
π/2
0
nω
Ψ POSIT. MAX., Ψ' = 0
Ψ = 0, Ψ' NEGAT. MIN.
Ψ NEGAT. MIN., Ψ' = 0
Ψ = 0, Ψ' POSIT. MAX.
Ψ POSIT. MAX., Ψ' = 0

Fig. 2.

The main motion occurs in radial direction. Fig. 2 gives schematically the horizontal velocity components for $n = 50$, $\eta = 4$. The E—W component is more pronounced than the N—S component. We see that there will be no vortices in the horizontal plane; similarly there are no vortices in planes vertical to the surface. The movements, which are stationary in a coordinate system fixed in space, are vibrations outwards and inwards and back and forth when considered in a coordinate system rotating with the sun.

9. *Conclusions.* — The movements thus found do not correspond to the vortices described by the movements of sunspot groups. It is interesting to realize, however, that we have been able to show that there is a thin layer around the sun in which certain vibrations are possible. The mass-velocities vanish at a depth not deep below the surface, while they remain finite at the surface. It will be necessary to fit the velocities themselves and the vibrations of the density and pressure to conditions prevailing in the solar atmosphere.

The negative result as regards vortices expected on the base of observations does not necessarily mean that movements of sunspots could not represent hydrodynamical vortices. In order to study the question further, we must make other assumptions concerning the value of ν. An interesting case will be the one in which the quantity λ goes through zero.

10. *Summary.* — The general equations of motion for the rotating sun have been investigated in order to find movements deviating from the rotational symmetry, which might occur in connection with the differential rotation. As a first trial, which was expected to be mathematically simple, the problem has been limited to stationary movements of relatively small angular diameter. It is then found that the movements considered may occur in a thin layer around the sun only. The movements are vibrations upwards and downwards and back and forth. The radial velocity component is the strongest, the N—S component the weakest.

I should like to express my gratitude to Prof. J. M. BURGERS for much helpful advice and to Dr. G. B. VAN ALBADA for clarifying discussions. Further I should like to thank the "Delftsche Hoogeschoolfonds" for a grant which made it possible for me to stay at Delft, and the I. A. U. "Commission for exchange of astronomers" for financial support from funds given by UNESCO.

The study was started while I worked in the Astrophysical Institute of Paris as "Chercheur" of the "Centre National de la Recherche Scientifique".

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Paleontology. — *Lepidocyclinae from Rembang (Java) with description of L. Wanneri n. sp.* By D. VAN DEN ABEELE. (Communicated by Prof. PH. H. KUENEN.)

(Communicated at the meeting of May 28, 1949.)

The collection on which this report is based was gathered by Prof. WANNER from the Rembang beds in the vicinity of the villages of Soemberan, Bringin and the oilfield of Gegoenceng, all to the south-east of Rembang. The collection contained molluscs and foraminifera. The former were treated by J. WANNER and E. HAHN [1]). The foraminifera were presented by Prof. WANNER to the Geological Institute of the University of Amsterdam. The author studied part of these foraminifera, especially the genus Lepidocyclina. In this report the same locality numbers will be used as in the above mentioned article.

The summary on East-Indian Lepidocyclinae by CAUDRI [2]), VAN DER VLERK's determinationtable [3]) and some original descriptions were used for the preparation of this article.

I am grateful to Prof. RUTTEN for his assistance in the determination.

The following Lepidocyclinae were met with:

A-forms	*L.*	*subradiata* DOUVILLÉ	
	L.	*vandervlerki* CAUDRI	
	L.	*papulifera* DOUVILLÉ	
B-forms	*L.*	*(Nephrolepidina)*	*angulosa* PROVALE
	L.	,,	*ferreroi* PROVALE
	L.	,,	*martini* SCHLUMBERGER
	L.	,,	*rutteni* VAN DER VLERK
	L.	,,	*multilobata* GERTH
	L.	*(Multilepidina)*	*luxurians* TOBLER
	L.	,,	*wanneri* nov. spec.

With WANNER we can subdivide the Neogene of Rembang as follows:

Karren-limestone and Globigerina-marl
Orbitoidal-limestone.

The samples of this collection are for the greater part from the Orbitoidal-limestone, the outcrop of which is found in the tops of the anticlines.

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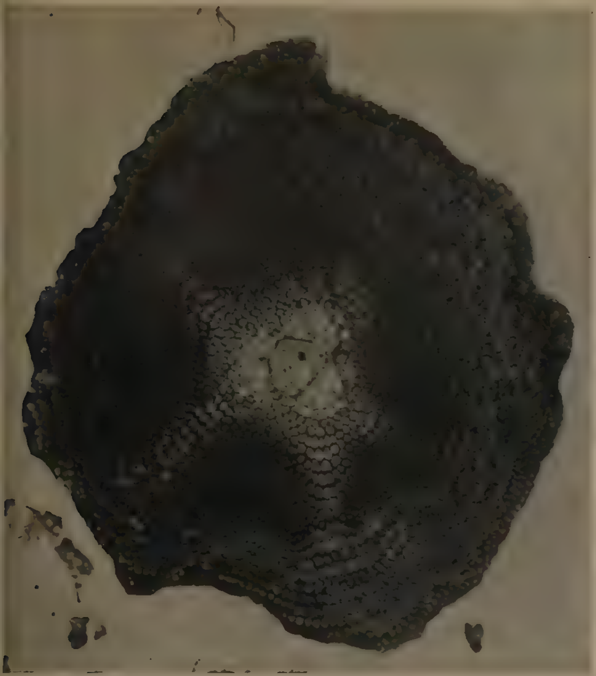
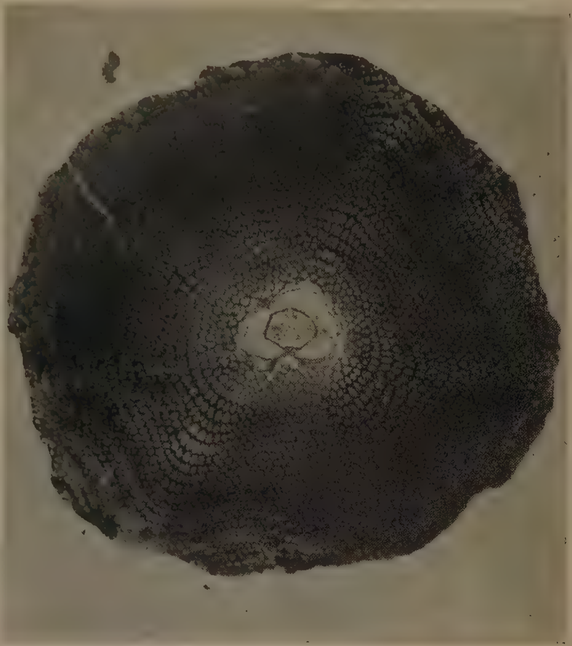
[1]) Miocäne Mollusken aus der Landschaft Rembang (Java). Zeitschr. d. Deutsch. Geol. Ges. 1935, p. 222.

[2]) Lepidocyclinen von Java. Verh. Geol. Mijnb. Gen. voor Ned. en Kol. ※, Deel XII, p. 135.

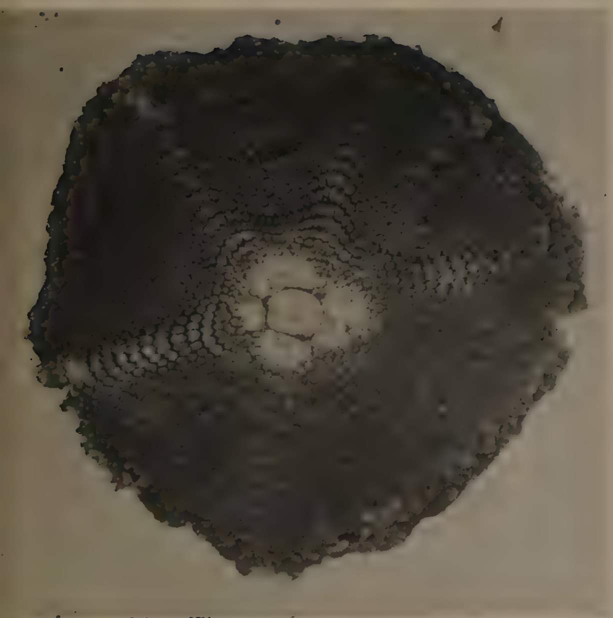
[3]) Het genus Lepidocyclina in het Indo-Pacifische gebied. Dienst van den Mijnbouw in Ned.-Indië no. 8, 1928.



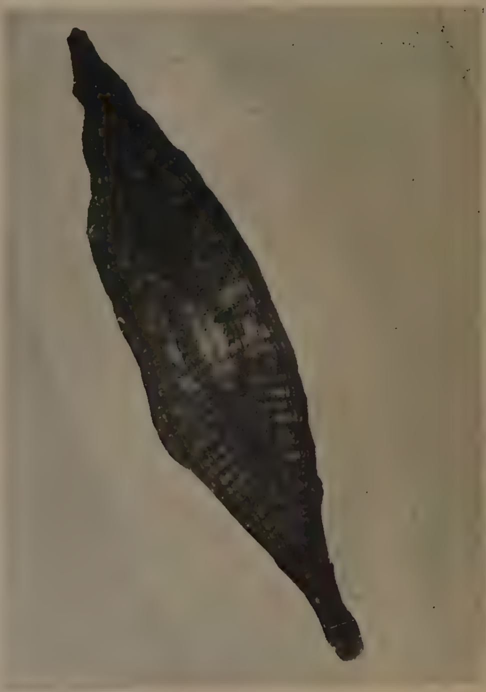
a.



b.



c.



d.

Lepidocyclina wanneri nov. spec. Three horizontal and one vertical section. 16 ×.

A list of localities shows the following distribution of samples in the two formations: (fig. 1, 2 and 3)

Fig. 1.

Fig. 2. Localities in the region of the Gegoenoeng-anticline. 1 : 100.000.

A. The Orbitoidal-limestone

Localities of the Gegoenoeng-anticline:

Sample 5. *L. borneensis* ?
Sample 6. *L. vandervlerki* ?
Sample 8. *L. subradiata*.
Sample 29. *L. subradiata, L. ferreroi, L. luxurians, L. wanneri*.
Sample 33. *L. subradiata, L. angulosa*.

Sample 33a *L. angulosa.*
Sample 275. L. *subradiata.*
Testpit I. *L. vandervlerki, L. martini.*

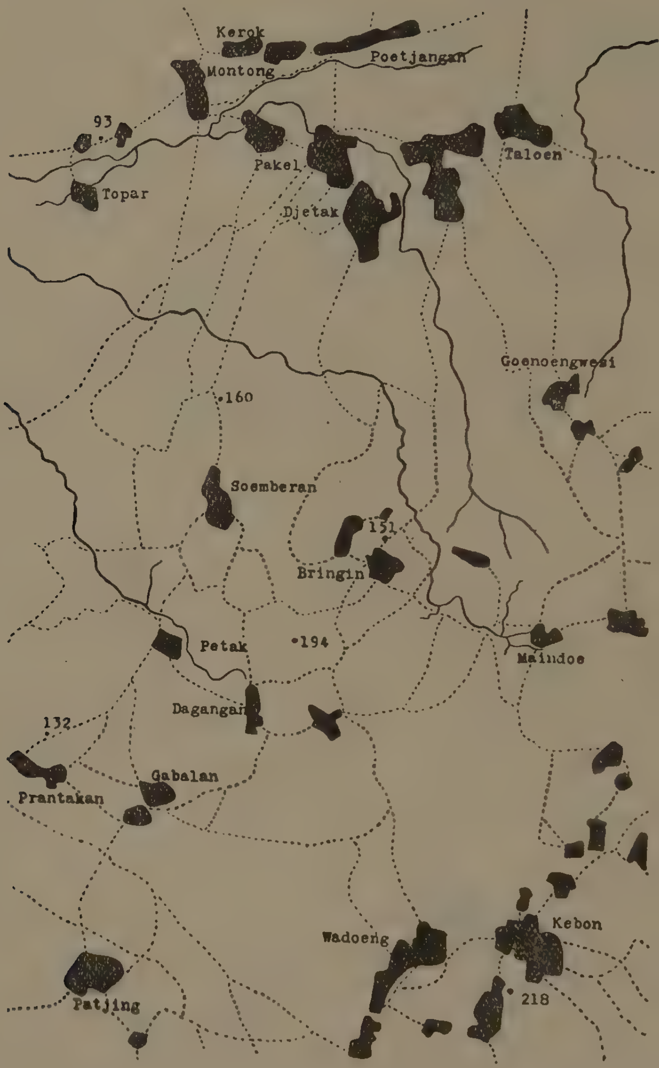


Fig. 3. Localities in the region of the Maindoe-anticline. 1 : 100.000.

Localities of the Bongkok-anticline:
Sample 17. *L. subradiata.*
Sample 24a. *L. subradiata.*
Sample 46. *L. subradiata.*
Sample 48. *L. martini.*
Sample 72. *L. papulifera.*
Sample 78. *L. subradiata.*
Sample 84. *L. subradiata.*

Tawoen-anticline:
Sample 64. *L. subradiata, L. multilobata.*

Localities of the Mahindoe-anticline:
Sample 132. *L. subradiata, L. angulosa, L. multilobata.*
Sample 93. *L. subradiata, L. angulosa, L. multilobata.*
Sample 151. *L. angulosa.*
Sample 160. *L. subradiata, L. multilobata, L. ferreroi, L. rutteni.*
Sample 194. *L. subradiata.*

B. The Globigerina-marl.

Sample 18. *L. angulosa.*
Sample 80. *L. angulosa.*
Sample 218. *L. luxurians.*

From locality 160 near Soemberan WANNER has collected 31 species of molluscs of which 5 or 16 % are still living. So the fauna must be classed as Old-Miocene, as used by MARTIN. The specimina of *Lepidocyclina rutteni* which occur here in the Lepidocyclina-series differ in some respects from those described by VAN DER VLERK, the variations are treated in the remarks below.

In sample 8 and Testpit I and II WANNER found 26 species of molluscs of which 12 % are still living. The *Lepidocyclinae* in these samples are: *L. vandervlerki, L. martini* and *L. angulosa.*

Description of the species.

Examination of the collection, showed that a multilepidine form occurs which could not be classed with one of the excisting species. For this form a new species was introduced.

In some other cases I thought it useful to add a supplementary remark to the description of CAUDRI; when nothing is mentioned the species under consideration fully agrees with her description.

Lepidocyclina subradiata DOUVILLÉ.

Specimina of this large microspheric species occur in many samples. The size varies widely, from 8 to 23 mm in diameter. Partly this will be due to a broken flange, but there are also small specimina that show no trace of being broken.

The specimina of sample 64 and 93 differ from those of the other localities by having well developed columns. The pillars have a diameter of 100—120 μ, are pentagonal or hexagonal in circumference and are separated from each other by one or two rings of lateral chambers. In size and shape of the test and in size and arrangement of the chambers there is no considerable difference with DOUVILLÉ's and CAUDRI's description.

The pillared specimina show affinity to both *L. subradiata* and *L. vandervlerki,* but on account of the small, rounded and thick-walled lateral chambers I have classified them as *L. subradiata.* The pillars suggest

L. vandervlerki, but here the lateral chambers have straight and thinner walls.

Lepidocyclina (Nephrolepidina) rutteni VAN DER VLERK.

Test rather small, not more than 5 mm in diameter and 2 mm thick in the center. A distinct central boss is surrounded by a wide, thin flange, which has been preserved in a few specimens only.

At every corner of the lateral chambers a pillar has developed, diameter of the chambers 150—300 μ and of the columns 70—150 μ.

The embryonic apparatus is of the nephrolepidine type or forms a transition towards the trybliolepidine form. The total diameter is 350—550 μ, the wall of the second chamber is $\pm$ 30 μ thick.

The equatorial chambers are arranged in polygons, towards the periphery the arrangement becomes vague as the polygons have been disturbed by many regenerations. The chambers are small, average size 50—75 μ, at the periphery they reach a maximum of 60 $\times$ 100 μ. The shape varies from spatulate to hexagonal.

The lateral chambers are arranged in regular tiers, 12—18 on each side of the equatorial plane. The length of the chambers over the center near the periphery is 230 μ, height 50 μ. The horizontal walls are straight or slightly curved.

Although in none of the slides a typical trybliolepidine nucleoconch is found, I would call these specimina *L. rutteni,* because in all other features they show close affinity to this species.

Lepidocyclina (Multilepidina) luxurians TOBLER.

There is a great difference in size between the specimina of the two localities where this species is found. In sample 29 the diameter does not exceed 2 mm, whereas in 219 it reaches 7 mm. In structure, however, there is great conformity, the nucleoconch is in both types very large, 1 mm in diameter. Further the large equatorial chambers of the first ring form a typical feature of this species.

In the small specimina we find only a few circular rings of chambers.

Lepidocyclina (Multilepidina) wanneri nov. spec. (Plate I and fig. 4)

Test discoidal, 3—5 mm in diameter, with a thickness in the center of 1½ mm. The circumference is slightly polygonal, almost circular, without any radiation. In the center of the test we find a number of large lateral chambers ($\pm$ 200 μ), the rest of the surface is formed by a regular pattern of considerably smaller chambers. The central pillars have an average diameter of 80 μ, they are surrounded by 4—6 chambers; near the periphery they are slightly smaller.

The embryonic apparatus is of the multilocular type: a central chamber surrounded by 2—4 semicircular chambers, that are only slightly smaller or equal in size. The initial chamber is 350—550 μ in diameter, the nucleoconch as a whole 800—900 μ. The walls are 15—40 μ thick.

Typical for this species is the polygonal arrangement of the equatorial chambers. In this respect it differs from the other Multilepidinae, like *L. luxurians* TOBLER (= *L. suvaensis* WHIPPLE?) and *L. stigteri* VAN DER VLERK.

These two species are characterized by a circular arrangement of their equatorial chambers, at most with a tendency to form polygons (WHIPPLE).

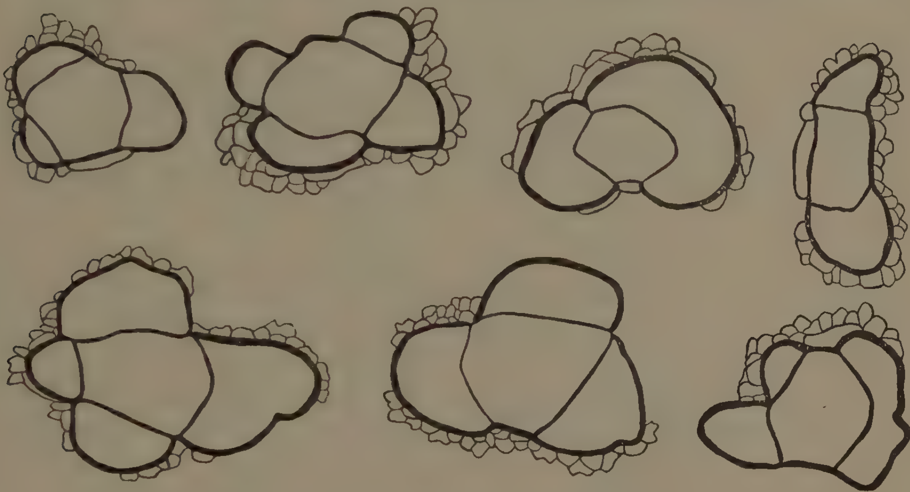


Fig. 4. Nucleoconch of Lepidocyclina wanneri, 40 ×.

The species under discussion, however, shows 4—7 distinct radii, in which the chambers are more elongate. In the first three or four cycles we find ogival or irregular shaped chambers; they are followed by spatulate to hexagonal chambers of rather a constant size. In the interradii this is 80 μ radial and 60 μ transverse, in the radii 100—120 μ and 60 μ. The thickness of the walls is 10—15 μ.

In the size of the equatorial chambers, too, this species differs from *L. luxurians* and *L. suvaensis*, where the chambers are slightly larger. In specimens of *L. luxurians* from this collection the chambers are not under 100 μ in radial diameter and often considerably larger.

In vertical section we see a regular arrangement of lateral chambers, 9—12 tiers on each side of the equatorial plane. In the center and near the periphery the length of the chambers is 250 μ, height 50 μ and the thickness of the walls about 15 μ. On either side of these central tiers there are much smaller chambers not exceeding 150 μ in length.

The horizontal walls of the lateral chambers are straight.

In the center the nucleoconch has a height of 60 μ, which in the radii increases to 100 μ.

This species occurs together with *L. subradiata, L. ferreroi* and *L. luxurians* in the same sample.

Number of specimens: 40.

Holotype no. C 1353. (Plate I. fig. c). Syntypes: C 1354—C 1388.

Amsterdam, May 1949. *Geological Institute.*

Geology. — *De vondsten bij het graven van de sluisput te Deventer 1948—1949.* (Voorlopige mededeling.) By J. BUTTER. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of June 25, 1949.)

1. Op 3 September 1942 werden door een baggermachine bij het graven en uitdiepen van de Nieuwe Haven te Deventer, onmiddellijk grenzende aan het terrein waar in 1948 en 1949 de sluisput gegraven zou worden, op ongeveer 6 m diepte opgebaggerd bij een waterstand van 1,38 m + N.A.P., dus op ongeveer 4,5 m — N.A.P. enige beenderen van

D 13 N.H. mammouth ⎫ bij elkaar behorende delen,
D 14 N.H. mammouth ⎭ van één been afkomstig;
D 20 N.H. rendiergewei, niet gerold, bijna geheel compleet;
D 19 N.H. horens van een groot rund;
D 24—29 N.H. geweistukken o.a. van Cervus Elaphus, en enkele stukken grint uit ditzelfde niveau, lengte 5, breedte 4½, hoogte 3; maar ook een paar grote stukken, 20 × 20 × 10 en 45 × 25 × 18.

2. In 1948 begon men op het terrein van de sluisput te Deventer voor het verlagen van de grondwaterspiegel 40 buizen te slaan, ± ½ m diameter, die tot ongeveer 26 m diep gingen.

Uit buis 9 kwam op September 1948 op ± 18,5 m diepte een stuk hertshoren te voorschijn (D 46 S).

Uit buis 12 kwam op 24/25 Augustus 1948 een stuk hout te voorschijn op ± 18—19 m diepte, dat de heer J. F. KOOLHAAS op het laboratorium van Prof. Dr E. REINDERS te Wageningen determineerde als zijnde van Ulmus.

3. In 1948 begon men met het maken van de sluisput, ongeveer 200 m lang en 25 m breed tot ongeveer 3,8 m — N.A.P. Het maaiveld lag op 6,5 m + N.A.P., vlak naast het reeds uitgebaggerde terrein van de Nieuwe Haven (zie de plattegrond, fig. 4).

Door het werken met excavateurs, die ± 1 m grond tegelijk in een bak scheppen, zijn er weinig beenderen te voorschijn gekomen, doordat ze meestal aan de aandacht van de arbeiders ontsnapten, geheel anders dan aan de Koerhuisbeek.

Op 14 November 1948 was bij Fig. 4, 1 het niveau van de put aldaar ongeveer 3,2 m — N.A.P.; ongeveer 2 dm dieper lag toen, dus op ± 3,4 m — N.A.P., een grintlaag.

Op die dag kwamen op ± *3,2 m — N.A.P. voor het eerst* enige schelpjes te voorschijn o.a. Puppa muscorum en 2 clausilia's, die de heer T. E. LOOSJES te Wageningen determineerde als te zijn Clausilia dubia en

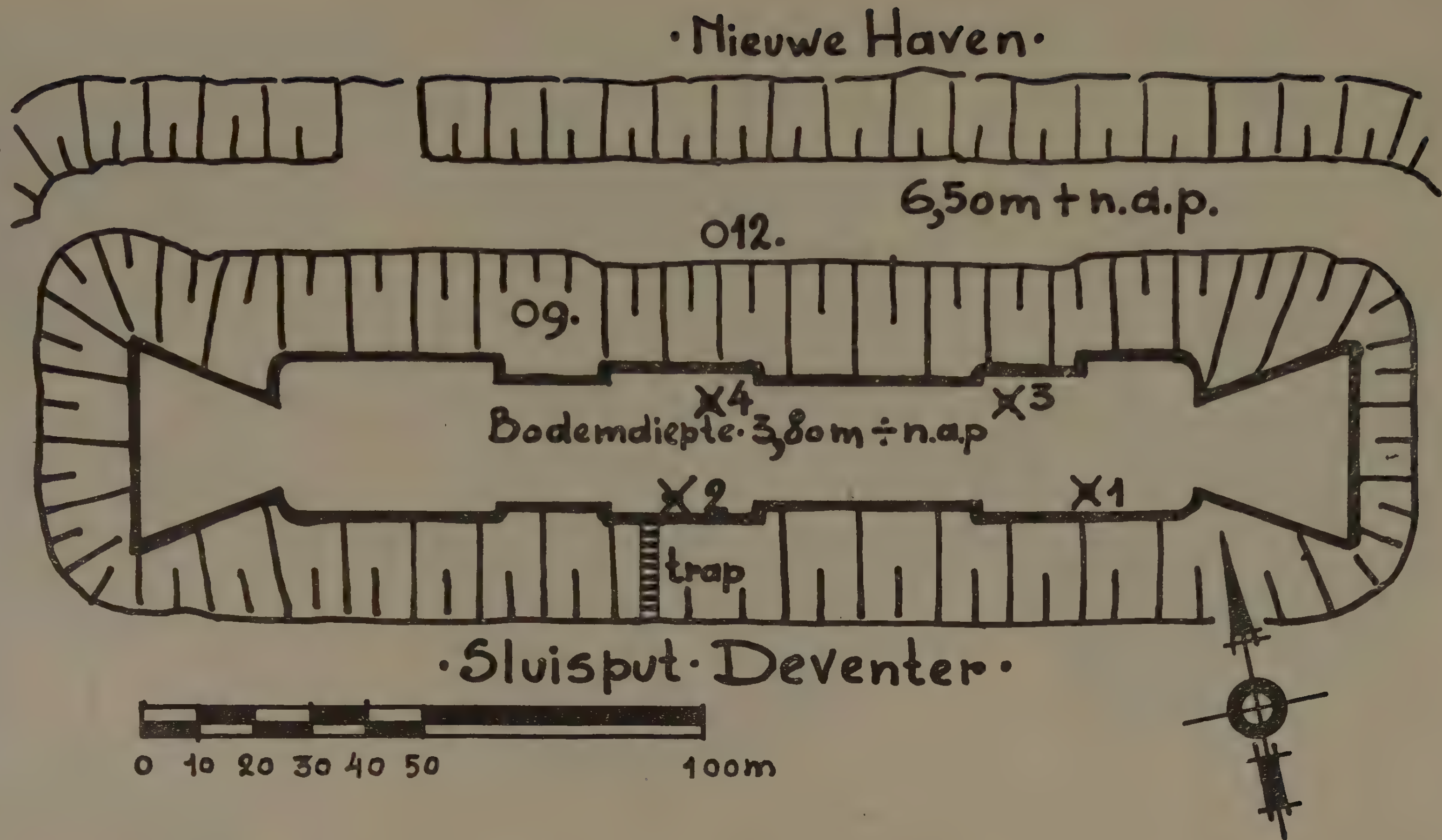
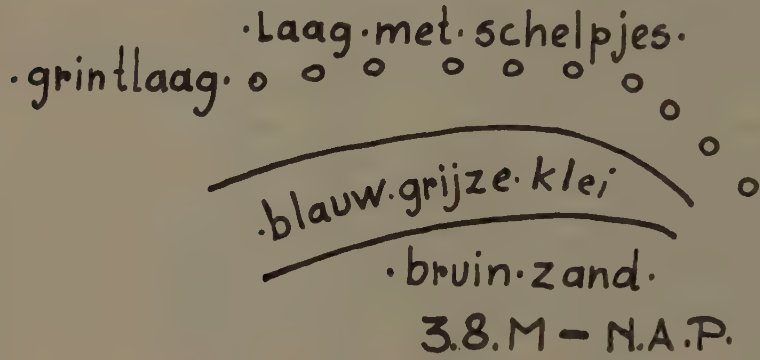


Fig. 4. Plattegrond van de Sluisput en Nieuwe Haven, naar gegevens van het Stedebouwkundig Advies- en Ingenieursbureau Irs WITTEVEEN en BOS, Deventer, getekend door A. HOOGENDOORN. 1, 2, 3, 4 zijn vindplaatsen van schelpjes boven de grintlaag. (Layers of shells; couches de coquilles.)

Clausilia plicatula (de laatste is voor het eerst gevonden in Nederland). De rest van de schelpjes en de nog later verzamelde, zijn gezonden aan mevrouw VAN DER FEEN te Amsterdam.

Op 1 Januari 1949 kwamen te voorschijn veel puppa's, cochlicopa's en andere zoetwatersoorten, verzameld bij Fig. 4, 2. Hier kwam de grintlaag, die bij 1 op 1 Januari 1949 was blootgekomen en daar bleek te liggen tussen 3.4 en 3,8 m — N.A.P. als een dunnere strook omhoog en er onder lag een gebogen laag blauwgrijze klei en daaronder bruin zand.



Hier was een rijke vindplaats van schelpen, circa 2 dm hoger dan de grintlaag, dus op 3,2 m — N.A.P. of iets hoger. Ook deze schelpjes zijn gezonden aan mevr. VAN DER FEEN. Ook bij 3 en 4 kwam een schelpjeslaag voor op hetzelfde niveau boven het grint.

Op 10 Januari 1949 ontving ik van de stortbaas WACHT, wonende te Epse (Gorssel):

A. Een stuk van een schedel van een mens, bestaande uit os occipitale en de beide os parietale. Vermoedelijk is het voorste stuk er afgebroken door de excavateur. Het schedeldeel heeft in geringe mate een „chignon". (Fig. 1.)

Hij had het Woensdag 5 Januari 1949 gevonden in aarde door de excavateurs aangebracht van Fig. 4, 1. De ontgraving was toen ongeveer op diepte, dus het is van dezelfde diepte als de schelpjeslaag boven de grintlaag op circa 3,20—3,4 m — N.A.P. De excavateurs brachten op diezelfde dag ook grint aan, wat hiermee in overeenstemming is. Het ziet er net zo van kleur uit als schedel C 2 van de Koerhuisbeek 1935/37.

B. Reeds enige tijd had de stortbaas in bezit 6 andere beenderen, die dus op geringer diepte vandaan komen van dezelfde plaats, n.l. een paar stukken gewei van C. elaphus, gebroken door de excavateur en een wervel met een lange doorn, vermoedelijk van C. megaceros?

Het grint uit de laag was gerold riviergrint met veel schuifstenen en alles zuidelijk materiaal.

$6 \times 5 \times 2\frac{1}{2}$, $3 \times 2\frac{1}{2} \times 2$, $1 \times 1 \times 1\frac{1}{2}$ en kleiner, 1 stuk $15 \times 10 \times 6$ en bevatte niet veel kwarts.

Het maakt de indruk van grintlaag III uit de Koerhuisbeek.

J. BUTTER: De vondsten bij het graven *van de sluisput te Deventer 1948—1949.*

De foto's zijn in opdracht van mij gemaakt door firma Hakeboom (1,2) en Koning (3).

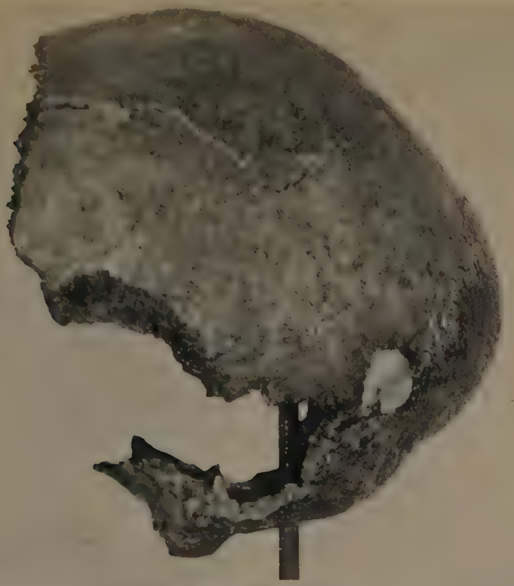


Fig. 1
Sluisput
Deventer
Schedel
S1 C1

Fig. 2
Sluisput
Deventer
S9 C2

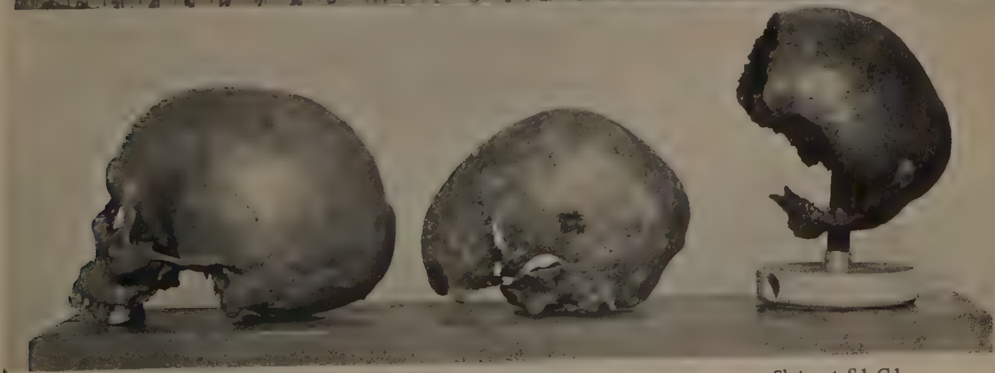


Fig. 3

Deventer
Koerhuisbeek C1
1935/37

Deventer
Koerhuisbeek C2
1935/37

Sluisput S1 C1
Deventer
1948/49

Andere grintsnoertjes waren slechts plaatselijk aanwezig, evenals enkele grijsblauwe kleilenzen, die aan de N.W. kant van de put zichtbaar waren. De rest was zand.

Maten bij mijn vele foto's zijn verstrekt door de heren LOK en LOHMANN en later C. F. SCHUURMAN van de uitvoerende Mij Bato, waarvoor hier mijn dank, evenals voor de belangstelling en medewerking van de Burgemeester Mr BLOEMERS en Irs WITTEVEEN en BOS, die me alle faciliteiten verschaften voor het betreden van het terrein en 't verschaffen van de vondsten.

Résumé:

1. sluisput: maaiveld 6,5 m + N.A.P.
2. sluisput: hertshorens, enz. in hogere lagen.
3. sluisput: schelpjes + schedel gedeelte circa 3,2 m — N.A.P.
4. sluisput: grintlaag circa 3,4—3,8 m — N.A.P.
5. Nieuwe Haven: Mammouth, rendier + rund + C. Elaphus, enz. ca. 4,5 m — N.A.P.

Het pleistocene niveau van mammouth en rendier werd in de sluisput niet bereikt, dus de schedel is niet pleistoceen.

De schedel S_1 C_1 lag boven het grint in het niveau van de schelpjeslaag, die veel landslakjes en zoetwaterslakjes bevatte, evenals aan de Koerhuisbeek en daarom ben ik ook geneigd deze schedel tot het eind van het boreale tijdvak (maglemose) te rekenen.

Op 21 Juni 1949 vond de werkman G. J. GARST, uit Terwolde, bij het graven voor een schutting dicht bij schelpenvindplaats 1 een schedel van een mens, die iets boven de grintlaag lag, welke hier ± 3 m — N.A.P. voorkwam. De schedel lag dus op ± 2,8 tot 2,9 m — N.A.P. De grintlaag was dezelfde als die van 14 November 1948 en liep dus iets omhoog, wat ook bij schelpenvindplaats 2 het geval was. De schedel heeft een schuin lopend voorhoofd en is dolichocephaal (Index ± 68). Fig. 2.

Een belangrijke conclusie is deze, dat van de Cromagnon-eigenschappen de Chignon lang blijft bestaan, waarvan vondsten van de Koerhuisbeek (1935—1937) en die van de sluisput (1948—1949) getuigen. Fig. 3.

Summary.

Sluicepit for the Nieuwe Haven (= new dock) of Deventer Netherlands (1948/49).

1. Surface sluicepit 6,5 + N.A.P. (N.A.P. = O.D.).
2. Sluicepit: in higher levels: horn of deer, etc.
3. Sluicepit ± 3,2 m — N.A.P.: shells (Clausilia dubia, Clausilia plicatula (Det. T. E. LOOSJES, Wageningen) Puppa muscorum, cochlicopa lubrica, freshwatershells and on nearly the same level: part of skull with os occipitale and 2 × os parietale. Fig. 4 near 1. Mrs VAN DER FEEN, Amsterdam, is determinating the other shells.

4. Sluicepit: gravel: 3,4—3,8 m — N.A.P.

5. New dock (Nieuwe Haven) (1942): Mammouth + horn of Rangifer-tarandus. Bos and Cervus elaphus ± 4.5 m — N.A.P. (from dredging machine).

Skull: Maximum hight 14.5 cm, maximum breadth 14 cm.

(Koerhuisbeek: 2 skulls, one with chignon.) Fig. 3.

Sluicepit: 1 part of skull with slight indication of "chignon. (Fig. 1)

The chignon is present till the boreal period (maglemose).

See Proc. Ned. Akad. v. Wetensch., Amsterdam, 1940. J. BUTTER, The excavation at Koerhuisbeek, Deventer, Netherlands.

Prof. Dr H. V. VALLOIS, Les ossements humains de Koerhuisbeek (idem 1943).

21 June 1949. Second skull (Fig. 2) 2.8—2.9 m — N.A.P., on gravel layer, near Fig. 4. 1. Dolichocephale (ind. ± 68) "front fuyant".

Résumé.

Les découvertes faites lors de l'exécution des fouilles en vue de la construction d'une écluse à Deventer (Province d'Overijssel, Pays-Bas) 1948—1949 et celles du creusement et l'approfondissement du nouveau port, dit „Nieuwe Haven", y attenant, 1942.

Conclusions des découvertes:

1. La côte du terrain naturel, au début des fouilles de l'emprise pour la construction de la nouvelle ecluse à Deventer 6.5 m + N.A.P. (N.A.P. = niveau de la mer à Amsterdam).

2. Dans les couches supérieures des fouilles en question: bois de cerf, etc.

3. A la côte env. 3.2 m — N.A.P. des coquilles terrestres, pupa muscorum, clausiliae, et coquilles d'eau douce (Fig. 4. 1.) et partie de crâne humain avec léger chignon.

4. A la côte 3.4 — N.A.P. jusqu'à 3.8 m — N.A.P. couche de gravier.

5. A la côte approchée 4.5 m — N.A.P. dans le *Nieuwe Haven*, y attenant, os de mammouth, bois de Rangifer tarandus, cervus elaphus et cornes de Bos, (1942) ramenés par une drague.

Dans les fouilles en vue de la construction de la nouvelle écluse, le niveau pléistocène, caractérisé par restes de mammouth et de renne, ne fut pas atteint.

Le crâne humain (os occipitale + 2 os parietale, max. hauteur 14½ cm, max. largeur 14 cm) (Fig. 1.) gisait à la côte, qui contenait beaucoup de coquilles d'espèces terrestres et d'eau douce, tout comme au Koerhuisbeek et c'est pourquoi je suis incliné à rattacher ce crâne humain à la fin de la période boréale (maglemosien) d'autant plus que ce crâne présente de grandes analogies avec ceux du Koerhuisbeek particulièrement en ce qui

concerne le chignon. (Pour les mesures des deux autre crânes de Koerhuisbeek. Comparez la littérature 2).

Une conclusion importante est celle que le caractère du chignon du crâne de Cromagnon persista longtemps jusque dans la période boréale ce dont témoignent les découvertes du Koerhuisbeek (1935—1937) et les trouvailles de l'emprise des fouilles de l'écluse de Deventer de 1948—1949. Comparez fig. 3.

21 juin 1949. Nouveau crâne humain, trouvé aux environs de gîte Fig. 4, 1.

La couche de gravier du 14 novembre 1948 se trouve içi à ± 3 m — N.A.P.

Le crâne se trouvait au-dessus de cette couche à 2,8—2,9 m — N.A.P.

Le front est fuyant et il est dolichocéphale (Index ± 68) (Fig. 2).

Schelpen uit de sluisput bij Deventer.

1ste zending, November 1948: (vindplaats Fig. 4, 1)

Neritina fluviatilis (L.) 1 ex.;
Valvata piscinalis (Müll.) 1 ex.;
Pupilla muscorum (L.) enige ex.

De eerste twee zijn zoetwaterdieren, de laatste een landslak. Bovendien bevonden zich in dit monster nog enkele fragmenten van oudere mollusken.

2de zending, Januari 1949: (hoofdzakelijk vindplaats Fig. 4, 2)

Neritina fluviatilis (L.) 1 ex.;
Valvata piscinalis (Müll.) vele ex.;
Bithynia tentaculata (L.) enige ex.;
Lymnaea palustris (Müll.) 1ex.;
„ ovata Drap. 1 ex.;
„ truncatula (Müll.) 2 ex.;
Planorbis planorbis (L.) enige ex.;
„ carinatus (Müll.) enkele ex.;
„ leucostoma (Müll.) 1 ex.;
Succinea oblonga Drap. verscheidene ex.;
Cochlicopa lubrica (Müll.) verscheidene ex.;
Pupilla muscorum (L.) vele ex.;
Vertigo pygmaea (Drap.) 1 ex.;
Vallonia pulchella (Müll.) enkele ex.;
Fruticicola hispida (L.) enige ex.;
Helicella erictorum (Müll.) 1 ex.;
Sphaerium solidum Norm. ½ ex.;
Pisidium amnicum (Müll.) 2/2 ex.

De eerste 9 en de laatste twee soorten zijn zoetwatermollusken, de tussenliggende 7 soorten zijn landslakken. Alle soorten komen ook recent voor; er is geen aanwijzing, dat een van alle een fossiele vorm van een recent nog voorkomende soort is.
Behalve de genoemde soorten bevat deze zending nog enkele fragmenten van oudere mariene mollusken.

Naschrift.

6 Juli 1949. Een derde schedel (S11 C3) wordt 2,8—2,9 m — N.A.P. gevonden bij Fig. 4, 1 in dezelfde schelpenlaag boven het grint.

6 July 1949. A third Skull (S11 C3) has been found (2,8—2,9 m — N.A.P.) at Fig. 4, 1 in the same layer with shells, above the gravel.

6 juillet 1949. Un troisième crâne (S11 C3) a été trouvé à Fig. 4, 1 (2,8—2,9 m — N.A.P.) dans la même couche avec des coquilles, immédiatement au-dessus du gravier.

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Zoology. — *Direct effects of isotonic and hypotonic lithium chloride solutions on unsegmented eggs of Limnaea stagnalis.* II. By CHR. P. RAVEN and J. R. ROBORGH. (Zoological Laboratory, University of Utrecht.)

(Communicated at the meeting of May 28, 1949.)

3. *Increase of amoeboid mobility.*

In normal development, the eggs show some amoeboid activity accompanying the maturation divisions (RAVEN 1945). Immediately after the extrusion of the first polar body, the outline of the egg becomes irregular, blunt pseudopodium-like processes are formed and a considerable deformation of the egg may occur, sometimes. This amoeboid activity reaches its maximum 5—10 minutes after the extrusion of the first polar body, then the egg rounds off again. The extrusion of the second polar body is accompanied again with a phase of amoeboid activity, which reaches its maximum about 10 minutes later, after which the outline of the egg becomes regular again. Both phases of amoeboid mobility correspond to drops in the tension at the surface, as determined by centrifugation. Considerable differences between egg-masses in the intensity of this amoeboid mobility occur.

DE GROOT (1948) observed abnormally strong amoeboid movements in eggs, which had been treated with 0.6—0.4 % solutions of LiCl and in which the formation of the second polar body was suppressed. It corresponded in time to the formation of the second polar body in the controls.

M. GRASVELD (1949) also observed this intense amoeboid mobility after Li-treatment. In 0.1 M (= 0.42 %) LiCl the eggs show these movements 3—7 hours after the first cleavage in the controls, in 0.05 M (= 0.21 %) after the extrusion of the first polar body and sometimes after the first cleavage.

In our experiments, this abnormally strong amoeboid mobility has also been observed in eggs treated with 0.20 % (isotonic) and 0.15—0.10 % (hypotonic) LiCl solutions and fixed after 2—3 hours. In all these eggs, the second maturation division has been finished, the first cleavage has (with few exceptions, cf. fig. 5*d*) not yet begun; the egg karyomeres and male pronucleus are situated near the animal pole. Whereas the control eggs are nearly spherical or show only slight deviations from a circular outline, many of the Li-treated eggs are very irregular in outline. Pseudopodium-like protuberances are formed especially at the animal side; they are of different sizes and may be pinched off nearly entirely by a deep furrow (fig. 5). In some of these eggs the karyomeres and pronucleus

lie at some distance from the animal pole in the interior cytoplasm; presumably, they are temporarily swept away from their normal position by the protoplasmic currents accompanying the formation of pseudopodia.

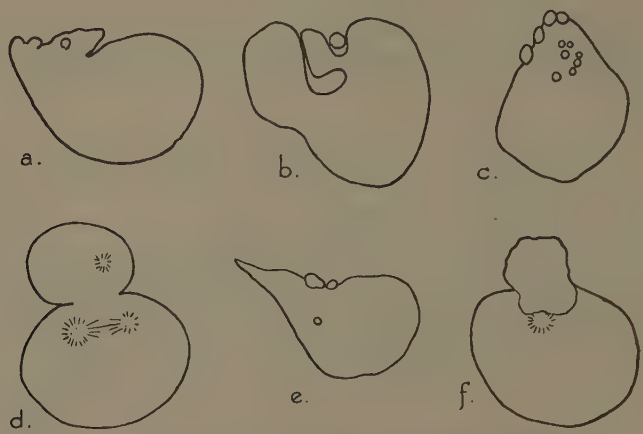

Fig. 5. Amoeboid mobility of Li-treated eggs.

In some batches, in which the spermaster is still visible in part of the eggs, the interesting observation has been made that a definite correlation exists between the presence of a (delayed) spermaster and the intensity of amoeboid movements. When the eggs of batches *B 6—2* and *B 4—3* (cf. p. 774/775) are divided into 3 groups, in which amoeboid activity is strong, moderate (slight deviations of circular outline) and weak (nearly circular outline), respectively, it appears that strong amoeboid activity is nearly restricted to the eggs in which the spermaster is still present (Table II and III).

This correlation is not an absolute one, however, for in other batches, in which the spermaster had disappeared in all eggs, still some highly amoeboid eggs have been found. Presumably, the same condition of the cytoplasm which delays the disappearance of the spermaster, promotes amoeboid mobility of the eggs.

TABLE II. *B 6—2.*

Spermaster \ Amoeb. activity	Strong	Moderate	Weak	
Present	11	3	—	14
Absent	—	14	2	16
	11	17	2	30

$\chi^2 = 20.073. \quad P = > 0.001$

TABLE III. B 4—3.

Spermaster \ Amoeb. activity	Strong	Moderate	Weak	
Present	8	3	1	12
Absent	2	5	4	11
	10	8	5	23

$\chi^2 = 5.867. \quad P = \pm 0.05$

4. *Karyomeres and pronuclei.*

In view of the fact that LiCl solutions at the 24-cell stage cause a swelling of the nuclei (RAVEN and DUDOK DE WIT 1949), and that a swelling of egg chromosomes and sperm nucleus immediately after the first maturation division in Li-treated eggs has been observed by us, one might expect a similar swelling of the nuclei to occur after the second maturation division. This is, however, not distinctly the case.

As stated above, after the extrusion of the second polar body the chromosomes remaining in the egg begin to swell into karyomeres and assemble beneath the egg cortex at the animal pole. The sperm nucleus, which in normal eggs begins its migration and transformation into the male pronucleus after the extrusion of the second polar body simultaneously with the swelling of the chromosomes, after some time reaches the animal pole, where it is found beside or beneath the egg karyomeres. Both egg karyomeres and male pronucleus enlarge still further by swelling. In the last hour before cleavage, the egg karyomeres begin to coalesce into a very irregular, polymorphic female pronucleus; in many cases this process is not yet entirely completed when the asters of the cleavage spindle appear.

We have studied a great number of controls and Li-treated eggs between the second maturation division and cleavage, and determined the sizes of karyomeres and male pronucleus. In most cases, however, no difference in nuclear size between both groups of eggs could be observed. Only in one batch, *D 2—3* (0.05 % LiCl, 3 hours), in which the coalescence of the karyomeres had advanced a great deal and which, therefore, must have been close before the onset of cleavage, the karyomeres and pronuclei were distinctly greater in the Li-treated eggs. In fact, they had reached gigantic dimensions: whereas the largest male pronucleus observed in the controls measured about $22 \times 12\,\mu$, in the treated eggs pronuclei up to $33 \times 30\,\mu$ were observed; hence, their volume had increased more than 9 times. In another batch (*C 2—4*, 0.10 % LiCl, 4 hours) it was noted that the pronuclei were greatly swollen; since the controls, in this case, all had begun to cleave, no comparison between controls and treated eggs could me made.

Whereas it is possible, therefore, that just before first cleavage LiCl has a swelling influence on the nuclei, during the greater part of the period between maturation and cleavage such an effect appears to be lacking.

5. *Delay of development.*

De Groot (1948) noted that cleavage may be delayed by treatment with LiCl. In LiCl 0.35 % and 0.30 % this delay is considerable; with decreasing concentration it decreases, and in 0.05 % there is no delay at all.

A delay of cleavage has also been observed by M. Grasveld (1949) in 0.2 % LiCl.

The same observation has been made by us in batches treated with 0.20 %—0.10 % LiCl; in 0.05 % LiCl no delay has been found. The first indication of it has been observed in some batches fixed after 3 hours, in which the coalescence of the egg karyomeres into a female pronucleus in the controls has advanced a great deal, whereas in the Li-treated eggs it is only just beginning. The delay of development becomes much more evident when cleavage begins. E.g. in batch *C 2—4* (0.10 %, 4 hours), of 28 controls 6 are in meta- or anaphase, 6 in telophase of first cleavage, whereas 16 are 2-cell stages; 12 of 15 Li-eggs, on the contrary, still have big swollen pronuclei, 3 are in pro-metaphase. Other batches show the same phenomenon.

6. *Abnormalities of cleavage mitosis.*

In normal development, first cleavage begins with the appearance of two small asters at opposite sides of the pronuclei, close to the nuclear membranes. Astral radiations extend a short distance into the cytoplasm, then, after the nuclear membranes near the asters are lost, the radiations quickly penetrate from both sides into the nuclear space, where they meet in the middle forming the central spindle. The chromatin of the pronuclei meanwhile condenses into the chromosomes, which are at first dispersed rather irregularly throughout the nuclear area, then gradually assemble near the equatorial region of the spindle, where they arrange themselves into the equatorial plate.

In the controls, the formation of the cleavage spindle proceeds in a normal way. In the Li-treated eggs, on the contrary, various abnormalities have been observed.

In batch *S—2* (0.15 % LiCl, 3 hours), among 12 eggs 3 show a beginning of spindle formation. One of the spindles is clearly tripolar, another shows indications of multipolarity.

In *S—3* (0.10 % LiCl, 3 hours; same egg-mass as *S—2*) in 3 out of 8 eggs spindle formation has begun; in all 3 cases the spindles are tripolar and possess 3 asters (fig. 6). In one, a big spherical cytoplasmic protuberance is present at the animal side, in which one of the 3 asters is

situated (fig. 5*d*); the chromatin has clumped together around the asters and on the spindle.

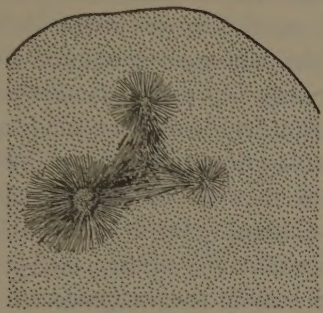

Fig. 6. *Limnaea stagnalis,* 0.10 % LiCl, 3 hours. Tripolar spindle of first cleavage.

In *D 1—4* (0.05 % LiCl, 4 hours), among 13 eggs 9 are in metaphase of first cleavage; 7 of these have a normal appearance, but in 2 the spindle has rotated 90° with respect to its normal position, so that its axis corresponds to the egg axis (fig. 7).

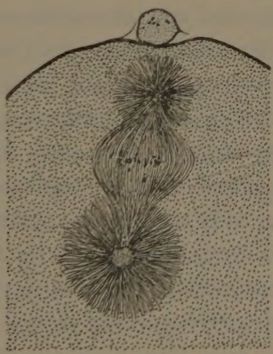

Fig. 7. *Limnaea stagnalis,* 0.05 % LiCl, 4 hours. First cleavage spindle with abnormal orientation.

We may conclude, therefore, that Li-treatment may lead to disturbances in the formation of the cleavage spindle and to changes in its position of equilibrium within the egg.

Discussion.

We may assume that the primary influence of the lithium ions penetrating the egg will be exercised on the cytoplasm. It is known that the small lithium ion, with its thick hydration layer, greatly affects the state of hydration of many gels. Hence, it seems likely that the lithium ions will influence the state of hydration of the protoplasmic colloids, thereby causing changes of water equilibrium between various components of the egg. Our results seem to be consistent with this view.

The condensation of the chromosomes during the prophase stage of mitosis may be considered as the expression of a changed relation of water equilibrium between the chromosomes and the cytoplasm, leading to a transfer of water from the former to the latter. On the other hand, the swelling of the telophase chromosomes into karyomeres or interphase nucleus at the end of mitosis represents the reverse process, whereby the original relation is restored. In the *Limnaea* egg, during the entire period between the first appearance of the tetrads in the ovarial or spermoviduct egg and the extrusion of the second polar body, apparently the water equilibrium is shifted consistently towards the side of the cytoplasm; the chromosomes of the egg and the sperm nucleus remain in a state of condensation. Immediately after the extrusion of the second polar body, a sudden change of water equilibrium occurs, leading to the swelling both of the egg chromosomes and the sperm nucleus, which in its turn brings about their mutual attraction and the migration of the sperm nucleus.

Our experiments have shown that the lithium ions influence this state of water equilibrium between chromosomes and cytoplasm. Under their action, this equilibrium is shifted towards the side of the chromosomes. This effect only becomes visible immediately after the extrusion of the first polar body, when it causes the egg chromosomes and the sperm nucleus to swell. We may assume that at this moment (telophase of first maturation division) also in the normal egg a slight shift in the water equilibrium occurs, corresponding to the processes occurring at this stage in ordinary mitosis, but which is insufficient, in this case, to bring about the swelling of the chromosomes. The added effect of the lithium ions, however, carries the equilibrium to the other side and causes, in this way, the chromosomes to swell. Shortly after, however, the deswelling actions, bound up with the beginning of the second maturation division, get the upper hand again and both egg chromosomes and sperm nucleus return to a state of condensation.

In normal development, from the extrusion of the second polar body until the beginning of cleavage, the water equilibrium between nuclei and cytoplasm consistently remains on the other side, the nuclei being in a state of swelling and even gradually increasing in size during this period. The lithium effect on the nuclei makes itself not perceptible during this phase; only at its end, immediately preceding cleavage, an increased swelling of the pronuclei has been observed. With the beginning of first cleavage and the condensation of cleavage chromosomes, again the state of equilibrium turns over to the other side. One might expect that this turn-over be delayed by the swelling-promoting action of lithium. It is possible, indeed, that the observed delay of cleavage is due to this cause; on the other hand, it may be that it is a first sign of the toxic actions of lithium leading to a final arrest of development at the 4-cell stage.

Another water equilibrium which comes into play is that between the asters, especially their central area ("central body") and the surrounding cytoplasm. It is generally admitted that the increase in size of the central

area of the asters is due to water absorption, fluid being transported in a centripetal direction between the astral rays. The rate of growth of this area will thus be dependent on the relation between its water binding capacity and that of the surrounding cytoplasm.

In *Limnaea* eggs treated with lithium chloride, the spermaster (inner aster of the second maturation spindle) during and after the extrusion of the second polar body exhibits an increased hydration of its central area, expressing itself in a highly vacuolated structure in the sections, the protoplasmic ground substance being reduced to a reticulum of fine meshes between the vacuoles. This may be explained by a shift in the water equilibrium between the aster and the cytoplasm. At the same time, the astral radiations are much more pronounced than in the controls in which they are rapidly disappearing at this time; perhaps, the increased inflow of water exerts a preserving influence on this structure. When the growth of the central area stops, the astral rays also in this case become blurred and disappear; the highly hydrated centre, however, remains visible for a long time.

The increase of amoeboid mobility of the eggs treated with lithium chloride may also be explained by a change in the physical properties of the cytoplasm. Our results show that a correlation exists between the increased amoeboid activity and the hydration of the spermaster. Hence, it may be assumed that both the change in water equilibrium between spermaster and cytoplasm and the increased amoeboid activity are due to the same modification in the state of the cytoplasm. It is interesting to note, in this connexion, that the periods of amoeboid mobility in normal development, immediately after the extrusion of both polar bodies, coincide with a rapid increase of the central areas of the maturation asters, preceding, in the case of the first maturation aster, its transformation into the second polar spindle, and in the case of the second maturation aster (= spermaster) its final disappearance.

As regards, finally, the cleavage anomalies, no consistent explanation along these lines can be given at present. Too little is known about the factors governing the appearance of asters and the direction of spindles to account for the observed multipolarity and abnormal position of cleavage spindles under the influence of lithium chloride. Evidently, in process of time the deflections from the normal course of development, brought about by the shifted interrelations between the egg components, become increasingly severe. This will lead, eventually, to the complete disruption of the normal interactions and the arrest of development at the 4-cell stage.

Summary.

1. Freshly-laid eggs of *Limnaea stagnalis* have been treated, after decapsulation, with 0.2 %—0.05 % solutions of lithium chloride.

2. In these eggs, the telophase chromosomes of the first maturation

division swell into karyomeres. At the same time, the sperm nucleus swells and migrates towards the animal pole.

3. The spermaster shows an increased hydration after the second maturation division; its disappearance is delayed.

4. The amoeboid mobility of the eggs is considerably increased after the second maturation division. This increase is correlated with the presence of a hydrated spermaster.

5. The first cleavage is delayed.

6. Various abnormalities of the first cleavage mitosis occur.

7. The results indicate that the primary action of the lithium ions consists in a change of the state of hydration of the protoplasmic colloids, causing changes of water equilibrium between various components of the egg.

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